Tensor Products of Principal Unitary Representations of Quantum Lorentz Group and Askey-Wilson Polynomials

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Abstract

We study the tensor product of principal unitary representations of the quantum Lorentz Group, prove a decomposition theorem and compute the associated intertwiners. We show that these intertwiners can be expressed in terms of complex continuations of $6j$ symbols of $\mathfrak{U}_q(su(2))$. These intertwiners are expressed in terms of q-Racah polynomials and Askey-Wilson polynomials. The orthogonality of these intertwiners imply some relation mixing these two families of polynomials. The simplest of these relations is the orthogonality of Askey-Wilson Polynomials.
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1 Introduction

In [1] we have pursued the work of Podles-Woronowicz [3] and Pusz [4]: we have shown that the unitary representations of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ can be nicely expressed in terms of one variable complex continuation of $6j$ symbols of $\mathfrak{U}_q(su(2))$. Using this result we were able to construct the characters of these unitary representations and prove a Plancherel theorem for L^2 functions. One has to work in the category of C^* multiplier Hopf algebras [3, 7] in order to handle functional analysis problems.

Let G be the complex Lie group $SL(2,\mathbb{C})$ and denote by $\{\overset{\lambda}{\Pi}, \lambda \in \frac{1}{2}\mathbb{Z}\times\mathbb{R}\}$ the set of principal unitary representations of G. The Plancherel measure is given by $P(\lambda)d\lambda = \frac{1}{2}(m^2+\rho^2)d\rho$ where $\lambda = (m,\rho)$ and $\overset{\lambda}{\Pi}$ is equivalent to $\overset{-\lambda}{\Pi}$. Naimark [10] has shown the following decomposition theorem:

$$\overset{(m,\rho)}{\Pi} \otimes \overset{(m',\rho')}{\Pi} = \bigoplus_{m''\in J_{m,m'}} \int^{\oplus} d\rho'' \overset{(m'',\rho'')}{\Pi}$$

where $J_{m,m'} = \{m \in \frac{1}{2}\mathbb{Z}, m+m'+m'' \in \mathbb{Z}\}$. The aim of the present article is to prove the quantum analog of this theorem and to give explicit formulae for the Clebsch-Gordan coefficients associated to this decomposition.

Let us also denote by $\{\overset{\lambda}{\Pi}, \lambda \in \frac{1}{2}\mathbb{Z}\times]-\frac{\pi}{\hbar},\frac{\pi}{\hbar}]\}$, where $q = e^{-\hbar}$, the set of principal representations of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$. The representation $\overset{\lambda}{\Pi}$ is a unitary representation of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ with domain V_λ. We have shown in [1], that these representations can be constructed using complex continuation in one variable (namely ρ) of $6j$ of $\mathfrak{U}_q(su(2))$. Let us denote by $6j(1)$ the complex continuation of these coefficients to distinguish them from the $6j$ of finite dimensional representations of $\mathfrak{U}_q(su(2))$, which will be denoted by $6j(0)$. We have shown that the Plancherel measure is $P(\lambda)d\lambda = (q-q^{-1})^2\frac{\hbar}{4\pi}[m+i\rho][m-i\rho]d\rho$ where $\lambda = (m,\rho)$.

In section 2 we recall definitions and properties of $6j(0)$ and $6j(1)$. We then introduce and study the basic properties of complex continuations, in three independent continuous spins, of $6j$ symbols that we call $6j(3)$. We review the main theorems of harmonic analysis on $SL_q(2,\mathbb{C})_{\mathbb{R}}$ which are needed in the sequel.

In section 3 we study the space of intertwiners $\Phi^{\lambda\lambda'}_{\lambda''} : V_\lambda \otimes V_{\lambda'} \to V_{\lambda''}$, and give an expression for them in terms of $6j(1)$ and $6j(3)$. Even in the classical case, such a simple expression of $\Phi^{\lambda\lambda'}_{\lambda''}$ was not known. We can define a linear map, denoted $\hat{\Phi}$, from $V_\lambda \otimes V_{\lambda'}$ to $\int^{\oplus} d\lambda'' \overset{\lambda''}{\Pi}$, by associating to each $u \in V_\lambda \otimes V_{\lambda'}$ the map $\lambda'' \mapsto \Phi^{\lambda\lambda'}_{\lambda''}(u)$. It remains to show that we can find a normalization $N(\lambda,\lambda',\lambda'')$ of $\Phi^{\lambda\lambda'}_{\lambda''}$ in such a way that $\hat{\Phi}$ is an isometry. This cannot easily be obtained from the definition of the $\Phi^{\lambda\lambda'}_{\lambda''}$ in terms of $6j(3)$.

We therefore use another construction of the intertwiners using the quantum analogue of the operator $\int dh(g)\, \overset{\lambda}{\Pi}(g) \otimes \overset{\lambda'}{\Pi}(g) \otimes \overset{\lambda''}{\Pi}(g^{-1})$ where dh is the Haar measure. By using the relations between these operators and the $\Phi^{\lambda\lambda'}_{\lambda''}$ we are able to compute the normalization factor. This is the content of section 4.

In section 5 we show that with this choice of normalization $\hat{\Phi}$ is an isometry. We then show that $\Phi^{\lambda\lambda'}_{\lambda''}$ is expressed in terms of $q-$ Racah polynomials and Askey-Wilson polynomials. As a result this last property implies non trivial identities which mix q-Racah polynomials and Askey-Wilson polynomials.

It is important to keep in mind the following hierarchy of complex continuations of $6j$ symbols of $\mathfrak{U}_q(su(2))$:

- $6j(0)$ are defined as being Racah coefficients (i.e recoupling coefficients) of finite dimensional representations of $\mathfrak{U}_q(su(2))$.
- $6j(1)$ are building blocks of matrix elements of unitary representations of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$. They are matrix elements of the universal shifted cocycle [13] and are equivalent to the Fusion Matrix of [14].
- $6j(3)$, as we will show, are building blocks of the Clebsch-Gordan coefficients associated to the tensor product of principal representations of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$. For the moment there is no real understanding of these $6j(3)$ in terms of Fusion matrix or as matrix elements of some universal element.

There also exists a final level of this hierarchy, called $6j(6)$, where the 6 spins are arbitrary complex numbers. They are the building blocks of the Racah coefficients of principal representations of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$. Their expressions in terms of basic hypergeometric functions, as well as their properties will be given in [2].

2 Definition and properties of various continuations of 6j symbols of $\mathfrak{U}_q(su(2))$

Let us first recall some notations and results of [1] which will be used throughout this work. The reader is also invited to read the first subsection of the appendix of the present article for definitions of basic hypergeometric functions.

2.1 Intertwiners and 6j(0) of $\mathfrak{U}_q(su(2))$

Let $q = e^{-\hbar}$ with $\hbar \in \mathbb{R}^{+*}$, $\mathfrak{U}_q(su(2))$ is the star Hopf algebra generated by $q^{\pm H}, J^{(\pm)}$ with the defining relations:

$$q^H J^{(\pm)} q^{-H} = q^{\pm 1} J^{(\pm)}, \qquad [J^{(+)}, J^{(-)}] = \frac{q^{2H} - q^{-2H}}{q - q^{-1}}, \qquad (J^{(\pm)})^* = q^{\mp 1} J^{(\mp)},$$
$$\Delta(J^{(\pm)}) = q^{-H} \otimes J^{(\pm)} + J^{(\pm)} \otimes q^H, \qquad \Delta(q^{\pm H}) = q^{\pm H} \otimes q^{\pm H}, \qquad (q^H)^* = q^H. \quad (1)$$

This Hopf algebra is a ribbon quasi-triangular Hopf algebra, with a universal $R-$matrix denoted R. We will define as usual $R^{(+)} = R, R^{(-)} = R_{21}^{-1}$ and $\mu = q^{2H}$.

We will denote in the rest of this article $Irr(\mathfrak{U}_q(su(2)))$ the set of all equivalence classes of finite dimensional irreducible unitary representations with $Sp(q^H) \in \mathbb{R}^+$. They are completely classified by a half-integer K and we will denote by $\overset{K}{\pi}$ the representation of spin K. The tensor product of elements of $Irr(\mathfrak{U}_q(su(2)))$ is completely reducible in elements of $Irr(\mathfrak{U}_q(su(2)))$. Let us define $\overset{K}{V}$ as being the vector space, of dimension $d_K = 2K+1$, associated to the representation of spin K. Let $(\overset{K}{e}_m)_{m=-K\cdots K}$, be an orthonormal basis of $\overset{K}{V}$ such that $\overset{K}{e}_m$ is of weight q^m for the action of q^H. The central ribbon element that we choose is such that $\overset{K}{\pi}(v) = v_K id$, where $v_K = e^{2i\pi K} q^{-2K(K+1)}$. Note that we have included a sign in order to satisfy relation (5).

Let us introduce the following notation: $\forall I, J, K \in \frac{1}{2}\mathbb{Z}^+, \forall m \in \frac{1}{2}\mathbb{Z}$ we define

$$Y^{(0)}_{(I,J,K)} = \begin{cases} 1 & \text{if} \quad I+J-K,\ J+K-I,\ K+I-J \in \mathbb{Z}^+ \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

$$Y^{(1)}_{(I,m)} = \begin{cases} 1 & \text{if} \quad I+m,\ I-m \in \mathbb{Z}^+ \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Let us now recall some properties about braiding matrices, $3j$ and $6j$ symbols of the finite dimensional representations of $\mathfrak{U}_q(su(2))$. We will always refer to conventions as well as explicit expressions given in [1], see also [15, 16, 17]. The group-like element $\overset{K}{\mu}$, the square root of the ribbon element $v_K^{1/2}$, the quantum Weyl element $\overset{K}{w}$, the braiding matrices $\overset{IJ}{R}{}^{(\pm)}$, and the Clebsch-Gordan coefficients satisfy the following relations:

$$\begin{aligned}
&\sum_{m,n}\left(\begin{array}{c|cc} r & I & J \\ L & m & n \end{array}\right)\left(\begin{array}{cc|c} m & n & K \\ I & J & p \end{array}\right)=Y^{(0)}_{(I,J,K)}\delta_{K,L}\delta^r_p Y^{(1)}_{(K,p)},\\
&\sum_{K,p}\left(\begin{array}{cc|c} m & n & K \\ I & J & p \end{array}\right)\left(\begin{array}{c|cc} p & I & J \\ K & i & j \end{array}\right)=\delta^m_i\delta^n_j Y^{(1)}_{(I,m)}Y^{(1)}_{(J,n)},
\end{aligned}\tag{4}$$

$$\sum_{k,l}\left(\begin{array}{c|cc} p & I & J \\ K & k & l \end{array}\right)\overset{JI}{R}{}^{(\pm)}{}^{lk}_{ji}=\left(\frac{v_I^{1/2}v_J^{1/2}}{v_K^{1/2}}\right)^{\pm1}\left(\begin{array}{c|cc} p & J & I \\ K & j & i \end{array}\right),\tag{5}$$

$$\sum_{i'j'm}\left(\begin{array}{c|cc} k & A & B \\ C & i' & j' \end{array}\right)\overset{AD}{R}{}^{(\pm)}{}^{i'l}_{im}\,\overset{BD}{R}{}^{(\pm)}{}^{j'm}_{jn}=\sum_p\overset{CD}{R}{}^{(\pm)}{}^{kl}_{pn}\left(\begin{array}{c|cc} p & A & B \\ C & i & j \end{array}\right),\tag{6}$$

$$\begin{aligned}
&\sum_{a'}v_I^{1/2}\,\overset{I}{w}_{aa'}\left(\begin{array}{cc|c} a' & b & K \\ I & J & k \end{array}\right)=e^{i\pi(J-K)}\left(\frac{[d_K]}{[d_J]}\right)^{1/2}\left(\begin{array}{c|cc} b & I & K \\ J & a & k \end{array}\right)=\sum_{k'}v_I^{1/2}\,\overset{K}{w}_{k'k}\left(\begin{array}{cc|c} b & k' & I \\ J & K & a \end{array}\right),\\
&\sum_{i',k'}\overset{J}{w}_{ii'}\,\overset{JI}{R}{}^{(\pm)}{}^{lk'}_{ji'}\,\overset{J}{w}{}^{k'k}=\overset{IJ}{R}{}^{(\mp)}{}^{il}_{kj}\,,\qquad\qquad v_J\sum_k\overset{J}{w}{}^{ik}\,\overset{J}{w}_{kj}=\delta^i_j Y^{(1)}_{(J,j)},
\end{aligned}\tag{7}$$

$$\left(\begin{array}{c|cc} 0 & I & J \\ 0 & a & b \end{array}\right)=\frac{\delta_{I,J}\,\overset{J}{w}_{ab}\,e^{i\pi J}}{\sqrt{[d_I]}},\qquad v_J\sum_b\overset{J}{w}{}^{ab}\,\overset{J}{w}_{kb}=e^{2i\pi J}\,\overset{J}{\mu}{}^a_{\ k},\qquad [d_K]=\sum_a\overset{K}{\mu}{}^a_{\ a},\tag{8}$$

$$\left(\begin{array}{c|cc} k & A & B \\ C & i & j \end{array}\right)=\left(\begin{array}{cc|c} i & j & C \\ A & B & k \end{array}\right)\in\mathbb{R}.\tag{9}$$

The Clebsch-Gordan coefficients also satisfy the relations [17]:

$$\left(\begin{array}{c|cc} k & A & B \\ C & i & j \end{array}\right)(q)=\left(\begin{array}{c|cc} -k & B & A \\ C & -j & -i \end{array}\right)(q)=(-1)^{A+B-C}\left(\begin{array}{c|cc} -k & A & B \\ C & -i & -j \end{array}\right)(q^{-1}).\tag{10}$$

Let us now recall basic facts about $6j(0)$ that we will use extensively in our work.
6j(0) of $\mathfrak{U}_q(su(2))$ are defined as follows:

$$\left\{\begin{array}{cc|c} A & B & E \\ C & F & D \end{array}\right\}_{(0)}\delta_{F,H}\delta^p_n Y^{(1)}_{(H,p)}=\sum_{i,j,k,l,m}\left(\begin{array}{c|cc} m & A & B \\ E & i & j \end{array}\right)\left(\begin{array}{c|cc} p & E & C \\ H & m & k \end{array}\right)\left(\begin{array}{cc|c} j & k & D \\ B & C & l \end{array}\right)\left(\begin{array}{cc|c} i & l & F \\ A & D & n \end{array}\right).$$

The properties of the Clebsch-Gordan coefficients, recalled above, imply the following relations on 6j(0):

$$\left\{\begin{array}{cc|c} A & B & E \\ C & D & F \end{array}\right\}_{(0)}=\left\{\begin{array}{cc|c} B & A & E \\ D & C & F \end{array}\right\}_{(0)}=\left\{\begin{array}{cc|c} C & D & E \\ A & B & F \end{array}\right\}_{(0)}=\left\{\begin{array}{cc|c} A & D & F \\ C & B & E \end{array}\right\}_{(0)}\ \text{(Symmetries)},\tag{11}$$

$$\left\{\begin{array}{cc|c} A & C & F \\ B & E & D \end{array}\right\}_{(0)}=e^{i\pi(C-F+E-D)}\left(\frac{[d_F][d_D]}{[d_C][d_E]}\right)^{1/2}\left\{\begin{array}{cc|c} A & F & C \\ B & D & E \end{array}\right\}_{(0)}\ \text{(Racah-Wigner)},\tag{12}$$

$$\sum_C \left\{ \begin{matrix} A & B \\ G & H \end{matrix} \middle| \begin{matrix} C \\ I \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} A & B \\ G & H \end{matrix} \middle| \begin{matrix} C \\ J \end{matrix} \right\}_{(0)} = \delta_{I,J} Y^{(0)}_{(A,H,I)} Y^{(0)}_{(B,G,I)} (\text{Orthogonality}), \tag{13}$$

$$\sum_C \left\{ \begin{matrix} A & B \\ H & G \end{matrix} \middle| \begin{matrix} C \\ I \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} A & B \\ G & H \end{matrix} \middle| \begin{matrix} C \\ J \end{matrix} \right\}_{(0)} \left(\frac{v_J^{1/2} v_I^{1/2} v_C^{1/2}}{v_G^{1/2} v_H^{1/2} v_A^{1/2} v_B^{1/2}} \right)^{\pm 1} = \left\{ \begin{matrix} A & G \\ B & H \end{matrix} \middle| \begin{matrix} I \\ J \end{matrix} \right\}_{(0)}, \tag{14}$$

$$\sum_A \left\{ \begin{matrix} D & F \\ I & G \end{matrix} \middle| \begin{matrix} A \\ J \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} D & F \\ E & B \end{matrix} \middle| \begin{matrix} A \\ C \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} E & A \\ G & H \end{matrix} \middle| \begin{matrix} B \\ I \end{matrix} \right\}_{(0)} = \left\{ \begin{matrix} E & F \\ J & H \end{matrix} \middle| \begin{matrix} C \\ I \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} D & C \\ H & G \end{matrix} \middle| \begin{matrix} B \\ J \end{matrix} \right\}_{(0)}. \tag{15}$$

The relation (14) is called the "Racah-relation", whereas the relation (15) is usually referred to as the "Pentagonal equation" or the "Biedenharn-Elliot" equation. The relations (14)(15) imply the Yang-Baxter equation on $6j(0)$, also called "Hexagonal relation".

Explicit expressions, as well as special values for low spins, of $6j(0)$ symbols are given in [1], but we want to add here the following asymptotic formulae which will be of importance in the rest of this paper (for details see [17]):

$$\lim_{K \to +\infty} \left\{ \begin{matrix} A & B \\ K & K+m+n \end{matrix} \middle| \begin{matrix} C \\ K+n \end{matrix} \right\}_{(0)} = e^{i\pi(A+B-C)} \left(\begin{matrix} m+n \\ C \end{matrix} \middle| \begin{matrix} A & B \\ m & n \end{matrix} \right), \tag{16}$$

$$\lim_{K \to +\infty} \left\{ \begin{matrix} A & K+n_1+n_2 \\ D & K \end{matrix} \middle| \begin{matrix} K+n_2 \\ K+n_1' \end{matrix} \right\}_{(0)} \left(\frac{v^{1/2}_{K+n_2} v^{1/2}_{K+n_1'}}{v_K^{1/2} v^{1/2}_{K+n_1+n_2}} \right)^{\pm 1} \delta_{n_1+n_2, n_1'+n_2'} = \overset{AD}{R}{}^{(\pm) n_1' n_2'}_{\ \ n_1 n_2}. \tag{17}$$

2.2 Definitions and properties of continuation of quantum $6j$ symbols

In order to understand the problem of continuation of quantum $6j$ symbols the reader is invited to read the appendix of [1] where definitions, proofs, as well as references on $6j(1)$ are given.

Let us give a definition which is essential in order to describe the complex continuations of quantum $6j$ symbols (the notations are explained in the appendix of the present work).

Definition 1 *Let* $\mathbb{X} = \{(T,U,V,X,Y,Z) \in \mathbb{C}^6, (2T, T+U-V, T+V-U, T+Y-Z, T+Z-Y) \in \mathbb{N}^5\}$, *we define for any* $(T,U,V,X,Y,Z) \in \mathbb{X}$,

$$\begin{aligned} \mathcal{N} \left(\begin{matrix} T & U \\ X & Y \end{matrix} \middle| \begin{matrix} V \\ Z \end{matrix} \right) &= e^{i\pi(T+Y-Z)}\, q^{(T+Y-Z)(U-T+Y-X+1)+(V+Y-X)(T+V-U)} \nu_1(d_Z) \nu_1(d_V) \times \\ &\times \frac{(U-T+X-Y+1)_\infty \ \omega(Y;V,X) \omega(T;Y,Z) \omega(T;V,U)}{(T-U+X+Y+1)_\infty (2T+1)_\infty (1)_\infty \ \omega(U;X,Z)} \end{aligned}$$

$$\left\{\!\!\left\{ \begin{matrix} T & U \\ X & Y \end{matrix} \middle| \begin{matrix} V \\ Z \end{matrix} \right\}\!\!\right\} = \mathcal{N} \left(\begin{matrix} T & U \\ X & Y \end{matrix} \middle| \begin{matrix} V \\ Z \end{matrix} \right) {}_4\Phi_3 \left[\begin{matrix} U-V-T & U+V-T+1 & Z-Y-T & -Z-Y-T-1 \\ & -2T & -Y-X-T+U & U+X-Y-T+1 \end{matrix} \right]. \tag{18}$$

It is important to understand that for $(T,U,V,X,Y,Z) \in \mathbb{X}$, $\mathcal{N} \left(\begin{matrix} T & U \\ X & Y \end{matrix} \middle| \begin{matrix} V \\ Z \end{matrix} \right)$ and $\left\{\!\!\left\{ \begin{matrix} T & U \\ X & Y \end{matrix} \middle| \begin{matrix} V \\ Z \end{matrix} \right\}\!\!\right\}$ are both square roots of rational functions in the variables $q^{2T}, q^{2U}, q^{2V}, q^{2X}, q^{2Y}, q^{2Z}$. This is a simple consequence of the fact that the hypergeometric series is of terminating type.

We have been very cautious with the determination of the signs. This annoying problem already appeared in the case of $6j(0)$ but can be handled quite easily. This problem is strengthned in the case of $6j(1)$ and $6j(3)$ because in that case we really have to take square roots of complex numbers. This is the reason why we introduced a particular square root, denoted $\nu_\infty(x)$ of the Eulerian product $(x)_\infty$ for $x \in \mathbb{C}$.

It will sometimes be useful to have the explicit value of the $6j$ symbols when one of the spins is equal to 0 or $\frac{1}{2}$. From the last definition we easily get:

$$\forall A,B,C\in\mathbb{C}, \left\{\!\!\left\{\begin{matrix} 0 & C \\ A & B \end{matrix}\middle| \begin{matrix} C \\ B \end{matrix}\right\}\!\!\right\} = 1 \tag{19}$$

$$\begin{aligned}
\left\{\!\!\left\{\begin{matrix} \frac{1}{2} & C+\frac{1}{2} \\ A & B \end{matrix}\middle| \begin{matrix} C \\ B+\frac{1}{2} \end{matrix}\right\}\!\!\right\} &= \frac{\nu_1(B+C-A+1)\nu_1(A+B+C+2)}{\nu_1(2C+2)\nu_1(2B+1)} \\
\left\{\!\!\left\{\begin{matrix} \frac{1}{2} & C+\frac{1}{2} \\ A & B \end{matrix}\middle| \begin{matrix} C \\ B-\frac{1}{2} \end{matrix}\right\}\!\!\right\} &= -\frac{\nu_1(A+B-C)\nu_1(A+C-B+1)}{\nu_1(2C+2)\nu_1(2B+1)}\, q^{C+B-A+1} \\
\left\{\!\!\left\{\begin{matrix} \frac{1}{2} & C-\frac{1}{2} \\ A & B \end{matrix}\middle| \begin{matrix} C \\ B+\frac{1}{2} \end{matrix}\right\}\!\!\right\} &= \frac{\nu_1(A+C-B)\nu_1(A+B-C+1)}{\nu_1(2C)\nu_1(2B+1)}\, q^{C+B-A} \\
\left\{\!\!\left\{\begin{matrix} \frac{1}{2} & C-\frac{1}{2} \\ A & B \end{matrix}\middle| \begin{matrix} C \\ B-\frac{1}{2} \end{matrix}\right\}\!\!\right\} &= \frac{\nu_1(A+B+C+1)\nu_1(C+B-A)}{\nu_1(2C)\nu_1(2B+1)}.
\end{aligned} \tag{20}$$

The usual $6j(0)$ symbols which properties were described in the last section and denoted $\left\{\begin{matrix} A & B \\ D & E \end{matrix}\middle| \begin{matrix} C \\ F \end{matrix}\right\}_{(0)}$ where $A,B,C,D,E,F\in\frac{1}{2}\mathbb{Z}^+$, are given by:

$$\left\{\begin{matrix} A & B \\ D & E \end{matrix}\middle| \begin{matrix} C \\ F \end{matrix}\right\}_{(0)} = \left\{\!\!\left\{\begin{matrix} A & B \\ D & E \end{matrix}\middle| \begin{matrix} C \\ F \end{matrix}\right\}\!\!\right\} Y^{(0)}_{(A,B,C)}Y^{(0)}_{(A,E,F)}Y^{(0)}_{(D,B,F)}Y^{(0)}_{(D,E,C).} \tag{21}$$

This expression can easily be obtained from the usual expressions [15, 16] using the inversion relation and the Sears identities (91,99) recalled in the appendix.

In the previous formula, and in the following ones, we will make a distinction between the first part of the right handside, which is called "explicit value", and the second part, products of Y functions and called "selection rules".

In order to describe properties of continuation of 6j symbols, we will make in the sequel a convenient abuse of notation, which greatly simplifies formulae:

if $X_1\in\mathbb{C}$ is fixed, if k is a positive integer and $f:\mathbb{C}^{k+1}\to\mathbb{C}$ is a function, a series of the type $\sum_{X_2,X_3,\dots,X_k} f(X_1,X_2,...,X_k)$ will always be defined as

$$\sum_{X_2,X_3,\dots,X_k} f(X_1,X_2,...,X_k) = \sum_{n_1,n_2,\dots,n_k\in\frac{1}{2}\mathbb{Z}} f(X_1,X_1+n_1,...,X_k+n_k). \tag{22}$$

Note that we will only use the notation X_0, X_1 to denote a couple of complex numbers such that $X_0=-\overline{X_1}-1$, and $X_0-X_1\in\frac{1}{2}\mathbb{Z}$, (this notation will be explained in the next part.)

Let us define the involutive endomorphism of the complex line: $\forall X\in\mathbb{C}, X\mapsto \underline{X}=-X-1$, as we will see it is important to understand the action of this symmetry on the $6j$ symbols.

In [18, 1] two types of $6j(1)$ symbols were defined. The first type is a family of numbers denoted $\left\{\begin{matrix} A & B \\ X_1 & X_2 \end{matrix}\middle| \begin{matrix} C \\ X_3 \end{matrix}\right\}_{(1)}$ and the second one is denoted $\left\{\begin{matrix} A & X_2 \\ B & X_1 \end{matrix}\middle| \begin{matrix} X_3 \\ X_4 \end{matrix}\right\}_{(1)}$, where both are defined for $A,B,C\in\frac{1}{2}\mathbb{Z}^+, \forall i,j\in\{1,\cdots,4\}, X_i\in\mathbb{C}-\frac{1}{2}\mathbb{Z}^+, X_i-X_j\in\frac{1}{2}\mathbb{Z}$.

Their explicit expressions are given by

$$\begin{aligned}
\left\{\begin{matrix} A & B \\ X_1 & X_2 \end{matrix}\middle| \begin{matrix} C \\ X_3 \end{matrix}\right\}_{(1)} &= \left\{\!\!\left\{\begin{matrix} A & B \\ X_1 & X_2 \end{matrix}\middle| \begin{matrix} C \\ X_3 \end{matrix}\right\}\!\!\right\} Y^{(1)}_{(A,X_2-X_3)}Y^{(1)}_{(B,X_1-X_3)}Y^{(1)}_{(C,X_1-X_2)}Y^{(0)}_{(A,B,C)}, \\
\left\{\begin{matrix} A & X_2 \\ B & X_1 \end{matrix}\middle| \begin{matrix} X_3 \\ X_4 \end{matrix}\right\}_{(1)} &= \left\{\!\!\left\{\begin{matrix} A & X_2 \\ B & X_1 \end{matrix}\middle| \begin{matrix} X_3 \\ X_4 \end{matrix}\right\}\!\!\right\} Y^{(1)}_{(A,X_2-X_3)}Y^{(1)}_{(A,X_1-X_4)}Y^{(1)}_{(B,X_1-X_3)}Y^{(1)}_{(B,X_2-X_4)}.
\end{aligned} \tag{23}$$

These formulae deserve some remarks, concerning the very word of "continuation". The explicit value of the $6j(1)$ is a regular function when the X_i approach half-integer values, but the selection rules appears to be different, i.e the support of the continuated 6j(1) are larger than that of the 6j(0) symbols. Nethertheless, for fixed $I, J \in \frac{1}{2}\mathbb{Z}^+$, and for a sufficiently large half-integer K we have $Y^{(1)}_{(I,J)} = Y^{(0)}_{(I,K,J+K)}$. It is in this sense that the term "continuation" to complex spins has been used.

From arguments developped in [18, 1], we can check different polynomial identities which are the continuation of the properties satisfied by the 6j(0). Rather than being exhaustive we will just mention some of them which will be important in our present work:

Proposition 1

$$\left\{\begin{array}{cc|c} A & B & C \\ X_1 & X_2 & X_3 \end{array}\right\}_{(1)} = \left\{\begin{array}{cc|c} B & A & C \\ X_2 & X_1 & X_3 \end{array}\right\}_{(1)} = (e^{i\pi}q)^{(X_1-X_3+A-C)} \frac{\nu_1(d_C)\nu_1(d_{X_3})}{\nu_1(d_A)\nu_1(d_{X_1})} \left\{\begin{array}{cc|c} C & B & A \\ X_3 & X_2 & X_1 \end{array}\right\}_{(1)},$$

$$\left\{\begin{array}{cc|c} A & X_2 & X_3 \\ B & X_1 & X_4 \end{array}\right\}_{(1)} = \left\{\begin{array}{cc|c} A & X_1 & X_4 \\ B & X_2 & X_3 \end{array}\right\}_{(1)} = \left\{\begin{array}{cc|c} B & X_1 & X_3 \\ A & X_2 & X_4 \end{array}\right\}_{(1)} =$$

$$= (e^{i\pi}q)^{(X_1-X_3+X_2-X_4)} \frac{\nu_1(d_{X_4})\nu_1(d_{X_3})}{\nu_1(d_{X_2})\nu_1(d_{X_1})} \left\{\begin{array}{cc|c} A & X_3 & X_2 \\ B & X_4 & X_1 \end{array}\right\}_{(1)}, \tag{24}$$

$$\sum_C \left\{\begin{array}{cc|c} A & B & C \\ X_1 & X_2 & X_3 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & B & C \\ X_1 & X_2 & X_4 \end{array}\right\}_{(1)} = \delta_{X_3,X_4} Y^{(1)}_{(A,X_2-X_3)} Y^{(1)}_{(B,X_1-X_3)}, \tag{25}$$

$$\sum_{X_3} \left\{\begin{array}{cc|c} A & B & C \\ X_1 & X_2 & X_3 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & B & D \\ X_1 & X_2 & X_3 \end{array}\right\}_{(1)} = \delta_{C,D} Y^{(0)}_{(A,B,C)} Y^{(1)}_{(C,X_1-X_2)}, \tag{26}$$

$$\sum_{X_3} \left\{\begin{array}{cc|c} A & X_2 & X_3 \\ B & X_1 & X_4 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & X_2 & X_3 \\ B & X_1 & X_5 \end{array}\right\}_{(1)} = \delta_{X_4,X_5} Y^{(1)}_{(A,X_1-X_4)} Y^{(1)}_{(B,X_2-X_4)}, \tag{27}$$

$$\sum_C \left\{\begin{array}{cc|c} A & B & C \\ X_2 & X_1 & X_3 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & B & C \\ X_1 & X_2 & X_4 \end{array}\right\}_{(1)} \left(\frac{v_{X_4}^{1/2} v_{X_3}^{1/2} v_C^{1/2}}{v_{X_1}^{1/2} v_{X_2}^{1/2} v_A^{1/2} v_B^{1/2}}\right)^{\pm 1} = \left\{\begin{array}{cc|c} A & X_1 & X_3 \\ B & X_2 & X_4 \end{array}\right\}_{(1)}, \tag{28}$$

$$\sum_A \left\{\begin{array}{cc|c} D & F & A \\ X_3 & X_1 & X_4 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} D & F & A \\ E & B & C \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} E & A & B \\ X_1 & X_2 & X_3 \end{array}\right\}_{(1)} = \left\{\begin{array}{cc|c} E & F & C \\ X_4 & X_2 & X_3 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} D & C & B \\ X_2 & X_1 & X_4 \end{array}\right\}_{(1)}, \tag{29}$$

$$\sum_{X_6} \left\{\begin{array}{cc|c} C & X_1 & X_4 \\ D & X_6 & X_2 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} C & X_2 & X_6 \\ A & X_5 & X_3 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & D & B \\ X_4 & X_5 & X_6 \end{array}\right\}_{(1)} = \left\{\begin{array}{cc|c} C & X_1 & X_4 \\ B & X_5 & X_3 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & D & B \\ X_1 & X_3 & X_2 \end{array}\right\}_{(1)}, \tag{30}$$

$$\sum_{X_5} \left\{\begin{array}{cc|c} B & A & P \\ X_2 & X_1 & X_5 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & Q & M \\ X_4 & X_5 & X_2 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} B & M & N \\ X_4 & X_1 & X_5 \end{array}\right\}_{(1)} = \left\{\begin{array}{cc|c} B & A & P \\ Q & N & M \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} P & Q & N \\ X_4 & X_1 & X_2 \end{array}\right\}_{(1)}, \tag{31}$$

$$\sum_M \left\{\begin{array}{cc|c} A & P & M \\ X_4 & X_3 & X_2 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & P & M \\ B & N & C \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} B & M & N \\ X_4 & X_1 & X_3 \end{array}\right\}_{(1)} \left(\frac{v_P^{1/2} v_{X_3}^{1/2}}{v_M^{1/2} v_{X_2}^{1/2}}\right)^{\pm 1} =$$

$$= \sum_{X_5} \left\{\begin{array}{cc|c} B & P & C \\ X_4 & X_5 & X_2 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & X_2 & X_3 \\ B & X_1 & X_5 \end{array}\right\}_{(1)} \left\{\begin{array}{cc|c} A & C & N \\ X_4 & X_1 & X_5 \end{array}\right\}_{(1)} \left(\frac{v_C^{1/2} v_{X_1}^{1/2}}{v_N^{1/2} v_{X_5}^{1/2}}\right)^{\pm 1}. \tag{32}$$

It is important to stress that every sums in the previous formulae are finite sums because of the selection rules entering into the definitions of $6j(1)$.

We also have the following symmetry:

Proposition 2

$$\left\{\begin{matrix} A & B \\ \underline{X_1} & \underline{X_2} \end{matrix}\middle|\begin{matrix} C \\ \underline{X_3} \end{matrix}\right\}_{(1)} = \left\{\begin{matrix} A & B \\ X_1 & X_2 \end{matrix}\middle|\begin{matrix} C \\ X_3 \end{matrix}\right\}_{(1)} (-1)^{A+B-C} \tag{33}$$

Proof:

From the Sears identity and the inversion relation (99, 91) we easily obtain the relation: $\left\{\begin{matrix} A & B \\ \underline{X_1} & \underline{X_2} \end{matrix}\middle|\begin{matrix} C \\ \underline{X_3} \end{matrix}\right\}_{(1)} = \left\{\begin{matrix} A & B \\ X_1 & X_2 \end{matrix}\middle|\begin{matrix} C \\ X_3 \end{matrix}\right\}_{(1)} f(A,B,C,X_1,X_1,X_3)$ where

$$f(A,B,C,X_1,X_1,X_3) = (-1)^{2A}\frac{\varphi(A+X_2+X_3,2A+1)\varphi(C+X_1+X_2,2C+1)}{\varphi(2X_3+1,1)\varphi(B+X_1+X_3,2B+1)}.$$

Using the explicit expression of $\phi(\alpha,n)$ for $n\in\mathbb{Z}$ explained in the appendix, we conclude that $f=(-1)^{A+B-C}$. $\square$

In order to obtain neat expressions for the Clebsch-Gordan coefficients of principal representations of the quantum Lorentz group, it is necessary to introduce a new level in this hierarchy of continuations of $6j$ symbols.

Definition 2 *We will define $6j(3)$ symbols to be the family of numbers denoted $\left\{\begin{matrix} A & X_1 \\ Z_1 & Y_1 \end{matrix}\middle|\begin{matrix} X_2 \\ Y_2 \end{matrix}\right\}_{(3)}$ where $A\in\frac{1}{2}\mathbb{Z}^+, X_1,X_2,Y_1,Y_2,Z_1\in\mathbb{C}-\frac{1}{2}\mathbb{Z}^+, X_1-X_2\in\frac{1}{2}\mathbb{Z}, Y_1-Y_2\in\frac{1}{2}\mathbb{Z}$ and defined by*

$$\left\{\begin{matrix} A & X_1 \\ Z_1 & Y_1 \end{matrix}\middle|\begin{matrix} X_2 \\ Y_2 \end{matrix}\right\}_{(3)} = \left\{\!\!\left\{\begin{matrix} A & X_1 \\ Z_1 & Y_1 \end{matrix}\middle|\begin{matrix} X_2 \\ Y_2 \end{matrix}\right\}\!\!\right\} Y^{(1)}_{(A,X_1-X_2)} Y^{(1)}_{(A,Y_1-Y_2)}. \tag{34}$$

These $6j(3)$ satisfy properties, which will be used in the rest of the paper and which are continuation of the identities satisfied by $6j(0)$ and $6j(1)$. We prefered to give combinatorial proofs of these properties rather than continuation arguments, in order to prepare to the $6j(6)$ case. Here again, the sums are finite which is a consequence of the selection rules in the definition of $6j(3)$. Note that these $6j(3)$ satisfy other relations, which will be explained in the section 5, as consequences of the study of the tensor product of principal representations of the quantum Lorentz group.

Proposition 3 *The following symmetry properties are satisfied:*

$$\left\{\begin{matrix} A & X_1 \\ Z_1 & Y_1 \end{matrix}\middle|\begin{matrix} X_2 \\ Y_2 \end{matrix}\right\}_{(3)} = \left\{\begin{matrix} A & Y_1 \\ Z_1 & X_1 \end{matrix}\middle|\begin{matrix} Y_2 \\ X_2 \end{matrix}\right\}_{(3)} = (e^{i\pi}q)^{(X_1-X_2+Y_1-Y_2)}\frac{\nu_1(d_{X_2})\nu_1(d_{Y_2})}{\nu_1(d_{X_1})\nu_1(d_{Y_1})}\left\{\begin{matrix} A & X_2 \\ Z_1 & Y_2 \end{matrix}\middle|\begin{matrix} X_1 \\ Y_1 \end{matrix}\right\}_{(3)}. \tag{35}$$

Proof:

Simple use of Sears identity (99). $\square$

Proposition 4 *We also have a discrete orthogonality property*

$$\sum_{X_2}\left\{\begin{matrix} A & X_1 \\ Z_1 & Y_1 \end{matrix}\middle|\begin{matrix} X_2 \\ Y_2 \end{matrix}\right\}_{(3)}\left\{\begin{matrix} A & X_1 \\ Z_1 & Y_1 \end{matrix}\middle|\begin{matrix} X_2 \\ Y_3 \end{matrix}\right\}_{(3)} = \delta_{Y_2,Y_3} Y^{(1)}_{(A,Y_1-Y_2)}. \tag{36}$$

Proof:

This is equivalent to the orthogonality of q-Racah polynomials (see our appendix for notations (101) and [19][20] for proofs):

$$\sum_{x=0}^{N} w^{(R)}(x;a,b,c,d)p_n^{(R)}(\mu(x);a,b,c,d)p_m^{(R)}(\mu(x);a,b,c,d) = \delta_{n,m}h_n^{(R)}(a,b,c,d), \tag{37}$$

$$\text{because} \quad \mathcal{N}\begin{pmatrix} T & X_1 & X_2 \\ Z_1 & Y_1 & Y_2 \end{pmatrix}^2 = \frac{w^{(R)}(x;a,b,c,d)}{h_n^{(R)}(a,b,c,d)}, \text{and}$$

$$\begin{aligned}
&{}_4\Phi_3\left[\begin{matrix} X_1-X_2-T & X_1+X_2-T+1 & Y_2-Y_1-T & -Y_2-Y_1-T-1 \\ -2T & -Y_1-Z_1-T+X_1 & X_1+Z_1-Y_1-T+1 & \end{matrix}\right] = p_n^{(R)}(\mu(x);a,b,c,d), \\
&\text{with} \quad n = T+X_2-X_1,\ x = T+Y_1-Y_2,\ N = 2T, \\
&a = q^{-4T-2},\ b = q^{4X_1+2},\ c = q^{2(Z_1+X_1-Y_1-T)},\ d = q^{-2(Z_1+Y_1+X_1+T+2)}.
\end{aligned} \tag{38}$$

□

These $6j(3)$ symbols satisfy the pentagonal relations:

Proposition 5

$$\sum_C \left\{\begin{matrix} A & B & C \\ Y_3 & Y_2 & Y_1 \end{matrix}\right\}_{(1)} \left\{\begin{matrix} B & A & C \\ X_1 & X_2 & X_3 \end{matrix}\right\}_{(1)} \left\{\begin{matrix} C & X_1 & X_2 \\ Z_1 & Y_3 & Y_2 \end{matrix}\right\}_{(3)} = \left\{\begin{matrix} A & X_1 & X_3 \\ Z_1 & Y_1 & Y_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} B & X_3 & X_2 \\ Z_1 & Y_3 & Y_1 \end{matrix}\right\}_{(3)}, \tag{39}$$

$$\sum_{X_3} \left\{\begin{matrix} B & X_2 & X_3 \\ Z_1 & Y_3 & Y_1 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} A & X_1 & X_2 \\ B & X_3 & X_4 \end{matrix}\right\}_{(1)} \left\{\begin{matrix} A & X_4 & X_3 \\ Y_3 & Z_1 & Z_2 \end{matrix}\right\}_{(3)} = \left\{\begin{matrix} A & X_1 & X_2 \\ Y_1 & Z_1 & Z_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} B & X_1 & X_4 \\ Z_2 & Y_3 & Y_1 \end{matrix}\right\}_{(3)}. \tag{40}$$

Proof:

These identities are simply proved by induction on A. Indeed, for $A = 0$ these identities are trivial and for $A = \frac{1}{2}$ they are easily checked on the exact expression of the $6j(3)$ using the finite difference equations verified by basic hypergeometric functions. The induction easily follows from the use of orthogonality and pentagonal equations on $6j(1)$.

□

Finally, we have a Racah relation:

Proposition 6

$$\sum_{Y_2} \left\{\begin{matrix} A & Y_1 & Y_2 \\ Z_2 & X_1 & X_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} A & Z_2 & Z_1 \\ X_1 & Y_1 & Y_2 \end{matrix}\right\}_{(3)} \left(\frac{v_{Y_2}^{1/2}v_{Z_1}^{1/2}v_{X_2}^{1/2}}{v_{Z_2}^{1/2}v_{Y_1}^{1/2}v_{X_1}^{1/2}v_A^{1/2}}\right)^{\pm 1} = \left\{\begin{matrix} A & X_1 & X_2 \\ Y_1 & Z_2 & Z_1 \end{matrix}\right\}_{(3)} \tag{41}$$

Proof:

It can be proved by continuation arguments from the $6j(1)$ case. □

We can prove other identities, which will be useful in the following chapters, and which are direct consequences of the previous ones.

Proposition 7 *The following pentagonal equations are satisfied:*

$$\sum_{X_2} \left\{\begin{matrix} C & X_1 & X_2 \\ Y_1 & Z_3 & Z_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} A & C & B \\ X_1 & X_3 & X_2 \end{matrix}\right\}_{(1)} \left\{\begin{matrix} A & X_2 & X_3 \\ Y_1 & Z_1 & Z_3 \end{matrix}\right\}_{(3)} = \left\{\begin{matrix} B & X_1 & X_3 \\ Y_1 & Z_1 & Z_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} C & A & B \\ Z_1 & Z_2 & Z_3 \end{matrix}\right\}_{(1)}, \tag{42}$$

$$\sum_{Z_2} \left\{\begin{matrix} A & X_4 & X_2 \\ Y_1 & Z_1 & Z_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} B & X_1 & X_4 \\ Z_2 & Y_1 & Y_2 \end{matrix}\right\}_{(3)} \left\{\begin{matrix} A & X_1 & X_3 \\ Y_2 & Z_1 & Z_2 \end{matrix}\right\}_{(3)} = \left\{\begin{matrix} A & X_1 & X_3 \\ B & X_2 & X_4 \end{matrix}\right\}_{(1)} \left\{\begin{matrix} B & X_3 & X_2 \\ Z_1 & Y_1 & Y_2 \end{matrix}\right\}_{(3)}. \tag{43}$$

Proof:
It is obtained from the pentagonal relations (39)(40) and the discrete orthogonality relation (36). □

We also have the hexagonal equation:

Proposition 8

$$\sum_C \frac{v_C^{1/2} v_{Y_2}^{1/2}}{v_B^{1/2} v_{Y_1}^{1/2}} \left\{ \begin{array}{cc|c} C & Y_3 & Y_1 \\ Z_1 & X_1 & X_2 \end{array} \right\}_{(3)} \left\{ \begin{array}{cc|c} A & B & C \\ Y_3 & Y_1 & Y_2 \end{array} \right\}_{(1)} \left\{ \begin{array}{cc|c} A & B & C \\ X_1 & X_2 & X_3 \end{array} \right\}_{(1)} =$$

$$= \sum_{Z_2} \frac{v_{X_2}^{1/2} v_{Z_2}^{1/2}}{v_{X_3}^{1/2} v_{Z_1}^{1/2}} \left\{ \begin{array}{cc|c} A & X_3 & X_2 \\ Y_3 & Z_1 & Z_2 \end{array} \right\}_{(3)} \left\{ \begin{array}{cc|c} A & Y_2 & Y_1 \\ X_1 & Z_1 & Z_2 \end{array} \right\}_{(3)} \left\{ \begin{array}{cc|c} B & X_1 & X_3 \\ Z_2 & Y_3 & Y_2 \end{array} \right\}_{(3)}. \quad (44)$$

Proof:
It is obtained by applying successively (41)(39)(43)(41). □

Finally the action of the automorphism of the complex line is given as follows:

Proposition 9

$$\left\{ \begin{array}{cc|c} A & \underline{X_1} & \underline{X_2} \\ \underline{Z_1} & \underline{Y_1} & \underline{Y_2} \end{array} \right\}_{(3)} = \left\{ \begin{array}{cc|c} A & X_1 & X_2 \\ Z_1 & Y_1 & Y_2 \end{array} \right\}_{(3)} g(A, X_1, X_2, Z_1, Y_1, Y_2)(-1)^{Y_1 - Y_2 + X_1 - X_2} \quad (45)$$

where g is a fourth root of unit which expression is:

$$g = \frac{\varphi(-Y_2+X_1+Z_1, X_1-X_2+Y_1-Y_2)\varphi(X_1+Y_2+Z_1+1, X_1-X_2+Y_2-Y_1)\varphi(X_1+Y_2-Z_1, X_1-X_2+Y_2-Y_1)}{\varphi(2Y_2+1,1)\varphi(2X_2+1,1)\varphi(Y_2+Y_1-A,-2A-1)\varphi(X_1+X_2-A,-2A-1)\varphi(Y_2+Z_1-X_1, Y_2-Y_1+X_2-X_1)}. \quad (46)$$

Proof:
It is proved using the inversion relation and the Sears identity. Using the explicit expression for $\varphi(\alpha, n)$ with n integer, we obtain that g is a fourth root of unit, but there is no simpler formula for g. □

2.3 Quantum Lorentz group, Principal Unitary Representations and Harmonic Analysis

We will recall in this section fondamental results on $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ and on the harmonic analysis [1] on $SL_q(2,C)_{\mathbb{R}}$.

$\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ by definition is the quantum double of $\mathfrak{U}_q(su(2))$. As a result we can write $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}}) = \mathfrak{U}_q(su(2)) \otimes \mathfrak{U}_q(su(2))^*$ as a vector space, where $\mathfrak{U}_q(su(2))^*$ denotes the restricted dual of $\mathfrak{U}_q(su(2))$, i.e the Hopf algebra spanned by the matrix elements of representations contained in $Irr(\mathfrak{U}_q(su(2)))$.

A basis of $\mathfrak{U}_q(su(2))^*$ is the set of matrix elements of irreducible representations of $\mathfrak{U}_q(su(2))$, which we will denote by $\overset{B}{g}{}^i_j, B \in \frac{1}{2}\mathbb{Z}^+, i, j = -B, ..., B$.

It can be shown that $\mathfrak{U}_q(su(2))^*$ is isomorphic, as a star Hopf algebra, to the quantum envelopping algebra $\mathfrak{U}_q(an(2))$ where $an(2)$ is the Lie algebra of traceless complex upper triangular 2×2 matrices with real diagonal.

$\mathfrak{U}_q(su(2))$ being a factorizable Hopf algebra, it is possible to give a nice generating family of $\mathfrak{U}_q(su(2))$. Let us introduce, for each $I \in \frac{1}{2}\mathbb{Z}^+$ the elements $\overset{I}{L}{}^{(\pm)} \in End(\mathbb{C}^{d_I}) \otimes \mathfrak{U}_q(su(2))$ defined by $\overset{I}{L}{}^{(\pm)} = (\overset{I}{\pi} \otimes id)(R^{(\pm)})$. The matrix elements of $\overset{I}{L}{}^{(\pm)}$ when I describes $\frac{1}{2}\mathbb{Z}^+$ span the vector space $\mathfrak{U}_q(su(2))$.

The star Hopf algebra structure on $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ is described in details in [1]. Let us simply recall that we have:

$$\overset{I}{L}{}^{(\pm)i}_{\ j}\overset{J}{L}{}^{(\pm)k}_{\ l} = \sum_{Kmn} \left(\begin{array}{cc|c} i & k & K \\ I & J & m \end{array} \right) \overset{K}{L}{}^{(\pm)m}_{\ n} \left(\begin{array}{c|cc} n & I & J \\ K & j & l \end{array} \right) , \qquad \overset{IJ}{R}_{12}\, \overset{I}{L}{}^{(+)}_1 \overset{J}{L}{}^{(-)}_2 = \overset{J}{L}{}^{(-)}_2 \overset{I}{L}{}^{(+)}_1\, \overset{IJ}{R}_{12}, \quad (47)$$

$$\overset{I}{g}{}^{i}_{j}\overset{J}{g}{}^{k}_{l} = \sum_{Kmn} \left(\begin{array}{cc|c} i & k & K \\ I & J & m \end{array} \right) \overset{K}{g}{}^{m}_{n} \left(\begin{array}{c|cc} n & I & J \\ K & j & l \end{array} \right) , \qquad \overset{IJ}{R}{}^{(\pm)}_{12} \overset{I}{L}{}^{(\pm)}_1 \overset{J}{g}_2 = \overset{J}{g}_2 \overset{I}{L}{}^{(\pm)}_1 \overset{IJ}{R}{}^{(\pm)}_{12}, \qquad (48)$$

$$\Delta(\overset{I}{L}{}^{(\pm)a}_{\ b}) = \sum_c \overset{I}{L}{}^{(\pm)c}_{\ b} \otimes \overset{I}{L}{}^{(\pm)a}_{\ c} , \qquad \Delta(\overset{I}{g}{}^{a}_{b}) = \sum_c \overset{I}{g}{}^{c}_{b} \otimes \overset{I}{g}{}^{a}_{c}, \qquad (49)$$

$$(\overset{I}{L}{}^{(\pm)a}_{\ b})^\star = S^{-1}(\overset{I}{L}{}^{(\mp)b}_{\ a}) , \qquad (\overset{I}{g}{}^{a}_{b})^\star = S^{-1}(\overset{I}{g}{}^{b}_{a}). \qquad (50)$$

The center of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ is a polynomial algebra in two variables Ω_+, Ω_- and we have $\Omega_\pm = tr(\overset{\frac{1}{2}}{\mu}{}^{-1}\overset{\frac{1}{2}}{L}{}^{(\mp)-1}\overset{\frac{1}{2}}{g})$.

Principal Unitary Representations of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ have been classified in [4], and the following description is given in [1]. They are labelled by a couple of complex numbers $(X_0, X_1) \in \mathbb{S}$, where $\mathbb{S} = \{(X_0, X_1) \in \mathbb{C}^2 \ / \ 2X_0{+}1 = (m{+}i\rho) \ , \ 2X_1{+}1 = (-m{+}i\rho), \quad m \in \frac{1}{2}\mathbb{Z}, \rho \in]-\frac{\pi}{\hbar}, \frac{\pi}{\hbar}] \ \}$. For $(X_0, X_1) \in \mathbb{S}$ we will often denote $m_X = X_0 - X_1, i\rho_X = X_0 + X_1 + 1$.

Let us denote by $\overset{(X_0X_1)}{\Pi}$ the principal representation associated to (X_0, X_1) and $V(X_0X_1)$ the associated $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ module. $V(X_0X_1)$ is a Harisch-Chandra module and we have $V(X_0X_1) = \bigoplus_{C, C-|m_x|\in\mathbb{N}} \overset{C}{V}$ as a $\mathfrak{U}_q(su(2))$ module. In term of the basis $\{\overset{C}{e}_r(X_0X_1) = \overset{C}{e}_r, C \in \frac{1}{2}\mathbb{Z}^+, r = -C, \ldots, C\}$ of the module $V(X_0X_1)$, the action of the generators of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ is given by:

$$\overset{B}{L}{}^{(\pm)i}_{\ j}\ \overset{C}{e}_r \ = \ \sum_n \overset{C}{e}_n\ \overset{BC}{R}{}^{(\pm)in}_{\ jr}, \qquad (51)$$

$$\overset{B}{g}{}^{i}_{j}\ \overset{C}{e}_r \ = \ \sum_{DEpx} \overset{E}{e}_p\ \left(\begin{array}{cc|c} p & i & D \\ E & B & x \end{array} \right) \left(\begin{array}{c|cc} x & B & C \\ D & j & r \end{array} \right) \Lambda^{BD}_{EC}(X_0X_1), \qquad (52)$$

where the complex numbers $\Lambda^{BD}_{EC}(X_0X_1)$ have to verify certain constraints explained in the appendix. It is a fondamental result that these coefficients can be expressed in terms of $6j(1)$ as follows:

$$\Lambda^{BC}_{AD}(X_0X_1) = \sum_{X_2} \left\{ \begin{array}{cc|c} B & C & A \\ X_0 & X_1 & X_2 \end{array} \right\}_{(1)} \left\{ \begin{array}{cc|c} B & C & D \\ X_0 & X_1 & X_2 \end{array} \right\}_{(1)} \frac{v_{X_2}}{v_{X_1}} \frac{v_A^{1/4} v_D^{1/4}}{v_B^{1/2} v_C^{1/2}} \qquad (53)$$

$$= \sum_{X_2} \left\{ \begin{array}{cc|c} B & C & A \\ X_1 & X_0 & X_2 \end{array} \right\}_{(1)} \left\{ \begin{array}{cc|c} B & C & D \\ X_1 & X_0 & X_2 \end{array} \right\}_{(1)} \frac{v_{X_0}}{v_{X_2}} \frac{v_B^{1/2} v_C^{1/2}}{v_A^{1/4} v_D^{1/4}}. \qquad (54)$$

The action of the center on the module $V(X_0X_1)$ is such that $\overset{(X_0X_1)}{\Pi}(\Omega_\pm) = \omega_\pm id$ where $\omega_+ = q^{2X_0+1} + q^{-2X_0-1}, \omega_- = q^{2X_1+1} + q^{-2X_1-1}$.

We can endow $V(X_0X_1)$ with a structure of pre-Hilbert space by defining the hermitian form $< .,. >$ such that the basis $\{\overset{C}{e}_r(X_0X_1), C \in \frac{1}{2}\mathbb{Z}^+, r = -C, \ldots, C\}$ of $V(X_0X_1)$ is orthonormal.

The representation $\overset{(X_0X_1)}{\Pi}$ is unitary in the sense that $\forall v, w \in V(X_0X_1), \ \ \forall a \in \mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$, $< a^* v, w > = < v, aw >$, this last property being equivalent to the relation: $\Lambda^{BC}_{AD}(X_1X_0) = \overline{\Lambda^{BC}_{AD}(X_0X_1)}$.

We will denote by $H(X_0X_1)$ the separable Hilbert space, completion of $V(X_0X_1)$ which Hilbertian basis is $\{\overset{C}{e}_r(X_0X_1), C\in\frac{1}{2}\mathbb{Z}^+, r=-C,\ldots,C\}$.

The automorphism of the complex line is now playing a key role because of the following important result: the principal representations associated to (X_0,X_1) and to $(\underline{X_0},\underline{X_1})$ are unitary equivalent.

Let us now recall some basic facts about the algebra of functions on $SL_q(2,\mathbb{C})_{\mathbb{R}}$ [3, 7, 1]. We will use the notations of [1]. The space of compact supported functions on the quantum Lorentz group, denoted $Fun_c(SL_q(2,\mathbb{C})_{\mathbb{R}})$ is, by definition, $Fun(SU_q(2)')\otimes\left(\bigoplus_{I\in\frac{1}{2}\mathbb{Z}^+}End(\mathbb{C}^{d_I})\right)$. This is a C^* algebra without unit. It contains the dense *-subalgebra [1] $Fun_{cc}(SL_q(2,\mathbb{C})_{\mathbb{R}}) = Pol(SU_q(2)')\otimes\left(\bigoplus_{I\in\frac{1}{2}\mathbb{Z}^+}End(\mathbb{C}^{d_I})\right)$ which is a multiplier Hopf algebra [6], and which can be understood as being the quantization of the algebra generated by polynomials functions on $SU(2)$ and compact supported functions on $AN(2)$.

$(\overset{C}{k}{}^m_n\otimes\overset{D}{E}{}^p_q)_{C,D,m,n,p,q}$ is a vector basis of $Fun_{cc}(SL_q(2,\mathbb{C})_{\mathbb{R}})$ which is defined, for example, by duality from the generators of the envelopping algebra:

$$<\overset{A}{L}{}^{(\pm)i}_{\ \ j}\otimes\overset{B}{g}{}^k_l,\ \overset{C}{k}{}^m_n\otimes\overset{D}{E}{}^p_q>=\overset{AC}{R}{}^{(\pm)im}_{\ \ jn}\delta_{B,D}\delta^k_q\delta^p_l. \tag{55}$$

We can describe completely the structure of the multiplier Hopf algebra in this basis;

$$\begin{aligned}
&\overset{A}{k}{}^i_j\otimes\overset{B}{E}{}^k_l\cdot\overset{C}{k}{}^m_n\otimes\overset{D}{E}{}^p_q=\sum_{Frs}\left(\begin{array}{cc|c} i & m & F\\ A & C & r\end{array}\right)\left(\begin{array}{c|cc} s & A & C\\ F & j & n\end{array}\right)\delta^p_l\delta_{B,D}\overset{F}{k}{}^r_s\otimes\overset{B}{E}{}^k_q\\
&\Delta(\overset{A}{k}{}^i_j\otimes\overset{B}{E}{}^k_l)=\mathcal{F}^{-1}_{23}\ (\sum_{\substack{C,D,m,\\ p,q,r,s}}\left(\begin{array}{cc|c} q & s & B\\ C & D & l\end{array}\right)\left(\begin{array}{c|cc} k & C & D\\ B & p & r\end{array}\right)\overset{A}{k}{}^i_m\otimes\overset{C}{E}{}^p_q\otimes\overset{A}{k}{}^m_j\otimes\overset{D}{E}{}^r_s\ )\ \mathcal{F}_{23}\\
&\epsilon(\overset{A}{k}{}^i_j\otimes\overset{B}{E}{}^k_l)=\delta^i_j\delta_{B,0}\qquad(\overset{A}{k}{}^i_j\otimes\overset{B}{E}{}^k_l)^\star=S^{-1}(\overset{A}{k}{}^j_i)\otimes\overset{B}{E}{}^l_k\quad\text{with}\ \ \mathcal{F}^{-1}_{12}=\sum_{J,x,y}\overset{J}{E}{}^x_y\otimes S^{-1}(\overset{J}{k}{}^y_x).
\end{aligned}\tag{56}$$

The space of right and left invariant linear forms (also called Haar measures) on $Fun_c(SL_q(2,\mathbb{C})_{\mathbb{R}})$ is a vector space of dimension one and we will pick one element h, which is defined by:

$$h(\overset{A}{k}{}^i_j\otimes\overset{B}{E}{}^m_l)=\delta_{A,0}(\overset{B}{\mu}{}^{-1})^m_l[d_B]. \tag{57}$$

Using the L^2 norm, $||\ a\ ||_{L^2}=h(a^*a)^{\frac{1}{2}}$, we can complete the space $Fun_c(SL_q(2,\mathbb{C})_{\mathbb{R}})$ into the Hilbert space of L^2 functions on the quantum Lorentz group, denoted $L^2(SL_q(2,\mathbb{C})_{\mathbb{R}})$.

$Fun_{cc}(SL_q(2,\mathbb{C})_{\mathbb{R}})$ is a multiplier Hopf algebra with basis $(u_\alpha)=((\overset{C}{k}{}^m_n\otimes\overset{D}{E}{}^p_q)_{C,D,m,n,p,q})$. The restricted dual of $Fun_{cc}(SL_q(2,\mathbb{C})_{\mathbb{R}})$, denoted $\tilde{\mathfrak{U}}_q(sl_q(2,\mathbb{C})_{\mathbb{R}})$, is the vector space spanned by the dual basis $(u^\alpha)=(\overset{C}{X}{}^n_m\otimes\overset{D}{g}{}^q_p)$. It is also, by duality, a multiplier Hopf algebra and $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ is included as an algebra in the multiplier algebra of $\tilde{\mathfrak{U}}_q(sl(2,\mathbb{C})_{\mathbb{R}})$. If Π is the principal representation of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$, acting on $V(Z_0Z_1)$, it is possible [1] to associate to it a unique representation $\tilde{\Pi}$ of $\tilde{\mathfrak{U}}_q(sl(2,\mathbb{C})_{\mathbb{R}})$, acting on $V(Z_0Z_1)$, such that $\tilde{\Pi}(\overset{C}{X}{}^n_m)(\overset{D}{e}_r)=\delta_{C,D}\delta^n_r\overset{D}{e}_m$.

We define for all ψ element of $Fun_{cc}(SL_q(2,\mathbb{C})_{\mathbb{R}})$, the operator $\Pi(\psi)=\sum_I\tilde{\Pi}(u^I)\ h(u_I\ \psi)$. It is easy to show that $\Pi(\psi)$ is of finite rank.

If f is a function on $\mathbb{S}$, we will sometimes write $f(m_X,\rho_X)$ instead of $f(X_0X_1)$. The Plancherel formula can be written as:

$$\forall\psi\in Fun_{cc}(SL_q(2,\mathbb{C})_{\mathbb{R}}),\ \ ||\ \psi\ ||^2_{L^2}=\int d\mathcal{P}(X_0X_1)\ \ \mathrm{tr}\left(\overset{(X_0X_1)}{\Pi}(\mu^{-1})\overset{(X_0X_1)}{\Pi}(\psi)\overset{(X_0X_1)}{\Pi}(\psi)^\dagger\right), \tag{58}$$

where we have denoted

$$\int d\mathcal{P}(X_0X_1)\ f(X_0X_1) = \sum_{m\in\frac{1}{2}\mathbb{Z}} \int_{-\frac{\pi}{\hbar}}^{\frac{\pi}{\hbar}} d\rho\ \ \mathcal{P}(m,\rho)\ f(m,\rho)$$

$$\text{with}\quad \mathcal{P}(m,\rho) = \frac{\hbar}{2\pi}(q-q^{-1})^2\ [m+i\rho][m-i\rho].$$

The proof of this theorem is purely combinatorial and uses as a central tool the following identity on Fourier coefficients of the Laurent polynomials Λ^{BC}_{AA}:

$$\int d\mathcal{P}(X_0X_1)\ \Lambda^{BC}_{AA}(X_0X_1)\ =\ \delta_{B,0}\delta_{A,C}[d_A]. \tag{59}$$

A Plancherel Theorem for L^2 functions has been proved [1], it follows easily from the previous result and from the following lemma, which will be useful later on

Lemma 1 *The only function $f:\mathbb{S}\to\mathbb{C}$ satisfying the following conditions:*
1)$\forall m\in\frac{1}{2}\mathbb{Z},\forall\rho\in]-\frac{\pi}{\hbar},\frac{\pi}{\hbar}],f(m,\rho)=f(-m,-\rho)$,
2)$\forall m\in\frac{1}{2}\mathbb{Z},f(m,.)$ is a L^2 function on $]-\frac{\pi}{\hbar},\frac{\pi}{\hbar}]$,
3)$\exists m_0\in\frac{1}{2}\mathbb{N},\forall m,|m|>m_0,f(m,.)=0$,
4) $\exists A,D\in\frac{1}{2}\mathbb{Z}^+\cap[|m_0|,+\infty[,\forall B,C\in\frac{1}{2}\mathbb{Z}^+,\ \int d\mathcal{P}(X_0X_1)\ f(X_0X_1)\ \Lambda^{BC}_{AD}(X_0X_1)\ =\ 0,$
is the nul function.

3 Intertwiners associated to unitary representations of the quantum Lorentz group

The aim of this section is to give explicit formulae of the intertwiners between the representation $\overset{(X_0X_1)}{\Pi}\otimes\overset{(Y_0Y_1)}{\Pi}$ and the representation $\overset{(Z_0Z_1)}{\Pi}$, in terms of complex continuations of $6j$ symbols of $\mathfrak{U}_q(su(2))$.

Proposition 10 *Let $\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}: V(X_0X_1)\otimes V(Y_0Y_1)\to V(Z_0Z_1)$ be a $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ intertwiner. We necessarily have:*

$$\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}(\overset{A}{e}_i\ (X_0X_1)\otimes\overset{B}{e}_j\ (Y_0Y_1)) = \sum_{C,k}\overset{C}{e}_k\ (Z_0Z_1)\left(\begin{array}{c|cc} k & A & B\\ C & i & j\end{array}\right)\left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & C\\ A & B & Z_0Z_1\end{array}\right], \tag{60}$$

where the coefficients $\left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & C\\ A & B & Z_0Z_1\end{array}\right]$, called "reduced elements", are complex numbers. Inversely such a map defines an intertwiner if and only if the following conditions on the reduced elements are satisfied: $\forall A,B,C\in\frac{1}{2}\mathbb{Z}^+, Y^{(0)}_{(A,B,C)}=Y^{(1)}_{(A,m_X)}=Y^{(1)}_{(B,m_Y)}=1$,

$$\sum_{QRSP}\left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & T\\ R & P & Z_0Z_1\end{array}\right]\Lambda^{US}_{RA}(X_0X_1)\Lambda^{UQ}_{PB}(Y_0Y_1)\left\{\begin{array}{cc|c} R & P & T\\ U & W & Q\end{array}\right\}_{(0)}\left\{\begin{array}{cc|c} B & U & Q\\ R & W & S\end{array}\right\}_{(0)}\times$$
$$\times\left\{\begin{array}{cc|c} U & A & S\\ B & W & C\end{array}\right\}_{(0)} = \left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & C\\ A & B & Z_0Z_1\end{array}\right]\Lambda^{UW}_{TC}(Z_0Z_1). \tag{61}$$

Proof:

The necessary condition comes from the fact that (60) is equivalent to the property that $\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ is an intertwiner of $\mathfrak{U}_q(su(2))$ module. Such a map is an intertwiner of $\mathfrak{U}_q(an(2))$ module, if moreover,

$$\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}(\Delta(\overset{U}{g}{}^m_n)(\overset{A}{e}_i\ (X_0X_1)\otimes \overset{B}{e}_j\ (Y_0Y_1))) = \overset{U}{g}{}^m_n \Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}(\overset{A}{e}_i\ (X_0X_1)\otimes \overset{B}{e}_j\ (Y_0Y_1)).$$

This last condition can be rewritten as (61) using (52)(25)(11). This concludes the proof. $\square$

Remark: A very important point is that $\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ maps the algebraic tensor product of the two domains $V(X_0X_1)\otimes V(Y_0Y_1)$ to $V(Z_0Z_1)$. As a result the sum (60) is finite. It can be seen that there are no non zero intertwiner from $V(Z_0Z_1)$ to the algebraic tensor product $V(X_0X_1)\otimes V(Y_0Y_1)$.

Lemma 2 *The space of* $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ *intertwiners from the module* $V(X_0X_1)\otimes V(Y_0Y_1)$ *to the module* $V(Z_0Z_1)$ *is of dimension* 0 *or* 1.

Proof:

An elementary proof is obtained by analyzing the rank of the system of linear equations (61). Using the isomorphism of module between $V(X_0X_1)$ and $V(\underline{X_0},\underline{X_1})$, we can always assume that m_X, m_Y, m_Z are non negative. It is easy to show that we can always assume that $m_Z \leq m_X + m_Y$. If not, we use the fact that the space of intertwiners from $V(X_0X_1)\otimes V(Y_0Y_1)$ to $V(Z_0Z_1)$ is in one to one correspondence with the space of intertwiners from $V(Z_0Z_1)\otimes V(\bar{X}_0\bar{X}_1)$ to $V(Y_0Y_1)$ to exchange m_Y and m_Z. This one to one correspondence easily follows from the isomorphism $V(X_0X_1)^* \approx \bar{V}(X_0X_1) \approx V(\bar{X}_0\bar{X}_1)$ and where we denoted by $V(X_0X_1)^*$ the restricted dual of the Harisch-Chandra module $V(X_0X_1)$.

This system of linear equations is equivalent to the subsystem where we have chosen $U=\frac{1}{2}$, because $\mathfrak{U}_q(an(2))$ is generated as an algebra by $\overset{U}{g}{}^m_n$ where $U=\frac{1}{2}$. This system is therefore equivalent to the following one: $\forall \sigma,\tau\in\{\frac{1}{2},-\frac{1}{2}\}$,

$$\sum_{\epsilon,\sigma,\mu,\tau\in\{-\frac{1}{2},\frac{1}{2}\}} \left[\begin{matrix} X_0X_1 & Y_0Y_1 \\ A+\nu+\rho & B+\epsilon+\mu \end{matrix}\middle|\begin{matrix} C+\sigma+\tau \\ Z_0Z_1\end{matrix}\right] \Lambda^{\frac{1}{2}\ \ A+\nu}_{A+\nu+\rho\ A}(X_0X_1)\Lambda^{\frac{1}{2}\ \ B+\epsilon}_{B+\epsilon+\mu\ B}(Y_0Y_1)\times$$
$$\times \left\{\begin{matrix} \frac{1}{2} & B+\epsilon+\mu \\ A+\nu+\rho & C+\sigma\end{matrix}\middle|\begin{matrix} B+\epsilon \\ C+\sigma+\tau\end{matrix}\right\}_{(0)} \left\{\begin{matrix} \frac{1}{2} & B \\ C+\sigma & A+\nu+\rho\end{matrix}\middle|\begin{matrix} B+\epsilon \\ A+\nu\end{matrix}\right\}_{(0)} \left\{\begin{matrix} \frac{1}{2} & A \\ B & C+\sigma\end{matrix}\middle|\begin{matrix} A+\nu \\ C\end{matrix}\right\}_{(0)} =$$
$$= \left[\begin{matrix} X_0X_1 & Y_0Y_1 \\ A & B\end{matrix}\middle|\begin{matrix} C \\ Z_0Z_1\end{matrix}\right]\Lambda^{\frac{1}{2}C+\sigma}_{C+\sigma+\tau C}(Z_0Z_1). \tag{62}$$

We will denote by $S(\sigma,\tau)$ this system of equations.

It is easy to show that the system $S(-\frac{1}{2},-\frac{1}{2})$ completely determines the reduced elements at the point (A,B,C) in terms of the reduced element of the points $(A',B',C-1)$ with $A'=A+\epsilon, B'=B+\nu$ with $\epsilon,\nu\in\{\frac{1}{2},-\frac{1}{2}\}$. Therefore the rank of the system is less than the rank of the vectors which components are the reduced elements at the points (A,B,m_Z).

Let $\Delta = \{(A,B)\in(\frac{1}{2}\mathbb{Z}^+)^2, Y^{(0)}_{(A,B,m_Z)}=1\}, P=\{(A,B)\in(\frac{1}{2}\mathbb{Z}^+)^2, A-m_X\in\mathbb{N}\}, Q=\{(A,B)\in\frac{1}{2}(\mathbb{Z}^+)^2, B-m_Y\in\mathbb{N}\}$. $\Delta\cap P\cap Q$ is the intersection of a lattice with a convex set which boundary consits in 3 segments and two half-lines. In the case where $m_Z\leq m_X+m_Y$ one of this segment degenerate to the point $p=(m_X,m_Y)$. It is easy to show, by a direct computation, that the system $S(\frac{1}{2},-\frac{1}{2})$ and $S(-\frac{1}{2},\frac{1}{2})$ are independent. As a result we can take linear combination of these two systems to have linear combinations of reduced elements at (A,B,m_Z) involving only 8 points and not 9. It is easy to show that the use of both of these systems determine uniquely the reduced elements at the point (A,B,m_Z) in terms of the reduced elements at the point $p, p+(1,0), p+(0,1), p+(1,1)$. As a result the system is of rank less than four. But the reduced elements at the point $p, p+(1,0), p+(0,1), p+(1,1)$ are solutions

of three systems of linear equations which can be shown to be independent:
$S(\frac{1}{2},-\frac{1}{2})$ at (m_X,m_Y,m_Z), $S(-\frac{1}{2},\frac{1}{2})$ at (m_X,m_Y,m_Z), and $S(\frac{1}{2},\frac{1}{2})$ at (m_X,m_Y,m_Z-1). As a result the rank of the system is of dimension less than one. $\square$

Theorem 1 *Assume that the numbers ρ_X,ρ_Y,ρ_Z and $\epsilon_X\rho_X+\epsilon_Y\rho_Y+\epsilon_Z\rho_Z$ with $\epsilon_X,\epsilon_Y,\epsilon_Z\in\{-1,1\}$ are non zero. The space of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ intertwiners from the module $V(X_0X_1)\otimes V(Y_0Y_1)$ to the module $V(Z_0Z_1)$ is of dimension 0 if and only if $m_X+m_Y+m_Z\notin\mathbb{Z}$.*
If $m_X+m_Y+m_Z\in\mathbb{Z}$ it is a one dimensional space which admits a non zero element, whose reduced element, given in terms of $6j(1)$ and $6j(3)$, is

$$\begin{bmatrix} X_0X_1 & Y_0Y_1 & T \\ R & P & Z_0Z_1\end{bmatrix}=\sum_{X_2}\begin{Bmatrix} T & Z_1 & Z_0 \\ Y_0 & X_0 & X_2\end{Bmatrix}_{(3)}\begin{Bmatrix} T & P & R \\ X_1 & X_0 & X_2\end{Bmatrix}_{(1)}\begin{Bmatrix} P & Y_1 & Y_0 \\ Z_1 & X_2 & X_1\end{Bmatrix}_{(3)}\frac{v_P^{1/4}v_T^{1/4}v_{X_0}^{1/4}v_{X_1}^{1/4}[d_P]^{1/2}}{v_R^{1/4}v_{X_2}^{1/2}e^{i\pi P}}. \tag{63}$$

We will denote by $\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ the associated intertwining operator.
The reduced elements satisfy the condition:
$$\forall C, C-|m_Z|\in\mathbb{N},\exists A,B\in\tfrac{1}{2}\mathbb{Z}^+,\begin{bmatrix} X_0X_1 & Y_0Y_1 & C \\ A & B & Z_0Z_1\end{bmatrix}\neq 0.$$

Proof:
We have assumed that ρ_X,ρ_Y,ρ_Z and $\epsilon_X\rho_X+\epsilon_Y\rho_Y+\epsilon_Z\rho_Z$ with $\epsilon_X,\epsilon_Y,\epsilon_Z\in\{-1,1\}$ are non zero in order that all the $6j(1)$ and $6j(3)$ are well defined in the expression (63). But this condition can nevertheless be removed by normalizing the reduced elements by a function $F(Z_0Z_1,Y_0Y_1,Z_0Z_1)$ which remove the singularities of this expression.

Using only polynomial identities on continuation of $6j$, we will show that the left handside of (61) and the righthandside of (61) are equal if we take the ansatz (63) for the reduced element.

The lefthandside of (61)=

$$=e^{-i\pi P}\sqrt{[d_P]}\sum_{QRSPX_2X_3Y_2}\frac{v_P^{1/2}v_T^{1/4}v_A^{1/4}v_B^{1/4}v_{X_0}^{1/4}v_{X_3}v_{Y_2}}{v_Uv_{X_2}^{1/2}v_S^{1/2}v_Q^{1/2}v_{X_1}^{3/4}v_{Y_1}}\begin{Bmatrix} T & Z_1 & Z_0 \\ Y_0 & X_0 & X_2\end{Bmatrix}_{(3)}\begin{Bmatrix} T & P & R \\ X_1 & X_0 & X_2\end{Bmatrix}_{(1)}$$
$$\times\begin{Bmatrix} P & Y_1 & Y_0 \\ Z_1 & X_2 & X_1\end{Bmatrix}_{(3)}\begin{Bmatrix} U & S & R \\ X_0 & X_1 & X_3\end{Bmatrix}_{(1)}\begin{Bmatrix} U & S & A \\ X_0 & X_1 & X_3\end{Bmatrix}_{(1)}\begin{Bmatrix} R & P & T \\ U & W & Q\end{Bmatrix}_{(0)}\begin{Bmatrix} B & U & Q \\ R & W & S\end{Bmatrix}_{(0)}$$
$$\times\begin{Bmatrix} U & A & S \\ B & W & V\end{Bmatrix}_{(0)}\begin{Bmatrix} U & Q & P \\ Y_0 & Y_1 & Y_2\end{Bmatrix}_{(1)}\begin{Bmatrix} U & Q & B \\ Y_0 & Y_1 & Y_2\end{Bmatrix}_{(1)}.$$

We apply first the pentagonal identity (31) to the second and sixth $6j$, then the pentagonal identity (31) to the fourth and the seventh $6j$ and finally we realize the sum on R by applying the orthogonality (25), to obtain

$$=e^{-i\pi P}\sqrt{[d_P]}\sum_{QSPX_2X_3Y_2X_4}\frac{v_P^{1/2}v_T^{1/4}v_A^{1/4}v_B^{1/4}v_{X_0}^{1/4}v_{X_3}v_{Y_2}}{v_Uv_{X_2}^{1/2}v_S^{1/2}v_Q^{1/2}v_{X_1}^{3/4}v_{Y_1}}\begin{Bmatrix} T & Z_1 & Z_0 \\ Y_0 & X_0 & X_2\end{Bmatrix}_{(3)}\begin{Bmatrix} U & P & Q \\ X_1 & X_4 & X_2\end{Bmatrix}_{(0)}$$
$$\times\begin{Bmatrix} P & Y_1 & Y_0 \\ Z_1 & X_2 & X_1\end{Bmatrix}_{(3)}\begin{Bmatrix} U & W & T \\ X_0 & X_2 & X_4\end{Bmatrix}_{(0)}\begin{Bmatrix} U & S & A \\ X_0 & X_1 & X_3\end{Bmatrix}_{(1)}\begin{Bmatrix} B & U & Q \\ X_1 & X_4 & X_3\end{Bmatrix}_{(1)}\begin{Bmatrix} B & W & S \\ X_0 & X_3 & X_4\end{Bmatrix}_{(1)}$$
$$\times\begin{Bmatrix} U & A & S \\ B & W & V\end{Bmatrix}_{(0)}\begin{Bmatrix} U & Q & P \\ Y_0 & Y_1 & Y_2\end{Bmatrix}_{(1)}\begin{Bmatrix} U & Q & B \\ Y_0 & Y_1 & Y_2\end{Bmatrix}_{(1)}.$$

Now we apply the symetries (24)(35) and transform the sum over P of the second, the third, and the ninth $6j$ using the hexagonal identity (44), to obtain

$$
\begin{aligned}
&= e^{-i\pi Q}\sqrt{[d_Q]}\sum_{QSZ_3X_2X_3Y_2X_4}\frac{v_T^{1/4}v_A^{1/4}v_B^{1/4}v_{X_0}^{1/4}v_{X_3}v_{Y_2}^{1/2}v_{Z_3}^{1/2}}{v_U v_{X_4}^{1/2}v_S^{1/2}v_{Z_1}^{1/2}v_{X_1}^{3/4}v_{Y_1}^{1/2}}\left\{\begin{array}{cc|c} T & Z_1 & Z_0 \\ Y_0 & X_0 & X_2 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & X_4 & X_2 \\ Y_0 & Z_1 & Z_3 \end{array}\right\}_{(3)} \\
&\times\left\{\begin{array}{cc|c} U & Y_2 & Y_1 \\ X_1 & Z_1 & Z_3 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & W & T \\ X_0 & X_2 & X_4 \end{array}\right\}_{(0)}\left\{\begin{array}{cc|c} U & S & A \\ X_0 & X_1 & X_3 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} B & U & Q \\ X_1 & X_4 & X_3 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} B & W & S \\ X_0 & X_3 & X_4 \end{array}\right\}_{(1)} \\
&\times\left\{\begin{array}{cc|c} U & A & S \\ B & W & V \end{array}\right\}_{(0)}\left\{\begin{array}{cc|c} Q & X_1 & X_4 \\ Z_3 & Y_0 & Y_2 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & Q & B \\ Y_0 & Y_1 & Y_2 \end{array}\right\}_{(1)}(e^{i\pi}q)^{(X_4+Y_1-X_1-Y_0)}\frac{\nu_1(d_{X_1})\nu_1(d_{Y_0})}{\nu_1(d_{X_4})\nu_1(d_{Y_1})}.
\end{aligned}
$$

We realize the sum over Q of the sixth, ninth and tenth $6j$ using the pentagonal identity (39) and symetries (24), the sum over X_2 of the first, second and fourth $6j$ using the pentagonal identity (39) and transform the sum over S of the fifth, seventh and eighth $6j$ according to the hexagonal identity (32) to obtain

$$
\begin{aligned}
&= e^{-i\pi B}\sqrt{[d_B]}\sum_{Z_3X_5X_3Y_2X_4}\frac{v_T^{1/4}v_V^{1/2}v_B^{1/4}v_{X_0}^{1/4}v_{X_3}^{1/2}v_{Y_2}^{1/2}v_{Z_3}^{1/2}}{v_W^{1/2}v_A^{1/4}v_U v_{X_5}^{1/2}v_{Z_1}^{1/2}v_{X_1}^{1/4}v_{Y_1}^{1/2}}\left\{\begin{array}{cc|c} U & Y_2 & Y_1 \\ X_1 & Z_1 & Z_3 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & X_1 & X_3 \\ Z_3 & Y_1 & Y_2 \end{array}\right\}_{(3)} \\
&\times\left\{\begin{array}{cc|c} B & X_3 & X_4 \\ Z_3 & Y_0 & Y_1 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} W & X_0 & X_4 \\ Y_0 & Z_3 & Z_0 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & W & T \\ Z_0 & Z_1 & Z_3 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} U & X_1 & X_3 \\ B & X_4 & X_5 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} B & V & A \\ X_0 & X_1 & X_5 \end{array}\right\}_{(1)} \\
&\times\left\{\begin{array}{cc|c} U & W & V \\ X_0 & X_5 & X_4 \end{array}\right\}_{(1)}(e^{i\pi}q)^{(X_4+2Y_1-X_1-Y_0-Y_2)}\frac{\nu_1(d_{X_1})\nu_1(d_{Y_0})\nu_1(d_{Y_2})}{\nu_1(d_{X_4})\nu_1(d_{Y_1})^2}.
\end{aligned}
$$

Then, we realize the sum over Y_2 of the first and second $6j$ by using the Racah identity (41) and symetries (35) to find

$$
\begin{aligned}
&= e^{-i\pi B}\sqrt{[d_B]}\sum_{Z_3X_5X_3X_4}\frac{v_T^{1/4}v_V^{1/2}v_{Z_3}v_B^{1/4}v_{X_0}^{1/4}v_{X_1}^{1/4}}{v_U^{1/2}v_A^{1/4}v_{Z_1}v_W^{1/2}v_{X_5}^{1/2}}\left\{\begin{array}{cc|c} U & X_1 & X_3 \\ Y_1 & Z_3 & Z_1 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} B & X_3 & X_4 \\ Z_3 & Y_0 & Y_1 \end{array}\right\}_{(3)} \\
&\times\left\{\begin{array}{cc|c} W & X_0 & X_4 \\ Y_0 & Z_3 & Z_0 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & W & T \\ Z_0 & Z_1 & Z_3 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} U & X_1 & X_3 \\ B & X_4 & X_5 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} B & V & A \\ X_0 & X_1 & X_5 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} U & W & V \\ X_0 & X_5 & X_4 \end{array}\right\}_{(1)} \\
&\times(e^{i\pi}q)^{(X_4+Y_1+Z_1-X_1-Y_0-Z_3)}\frac{\nu_1(d_{X_1})\nu_1(d_{Y_0})\nu_1(d_{Z_3})}{\nu_1(d_{X_4})\nu_1(d_{Y_1})\nu_1(d_{Z_1})}.
\end{aligned}
$$

Now, we realize the sum over X_3 of the first, second and fifth $6j$ by using the pentagonal identity (40) and symetries (35) to find

$$
\begin{aligned}
&= e^{-i\pi B}\sqrt{[d_B]}\sum_{Z_3X_5X_4}\frac{v_T^{1/4}v_V^{1/2}v_{Z_3}v_B^{1/4}v_{X_0}^{1/4}v_{X_1}^{1/4}}{v_U^{1/2}v_A^{1/4}v_{Z_1}v_W^{1/2}v_{X_5}^{1/2}}\left\{\begin{array}{cc|c} U & X_4 & X_5 \\ Y_0 & Z_1 & Z_3 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} B & X_5 & X_1 \\ Z_1 & Y_1 & Y_0 \end{array}\right\}_{(3)} \\
&\qquad\times\left\{\begin{array}{cc|c} W & X_0 & X_4 \\ Y_0 & Z_3 & Z_0 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} U & W & T \\ Z_0 & Z_1 & Z_3 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} B & V & A \\ X_0 & X_1 & X_5 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} U & W & V \\ X_0 & X_5 & X_4 \end{array}\right\}_{(1)}.
\end{aligned}
$$

Finally we realize the sum over X_4 of the first, third and sixth $6j$ using the pentagonal equation (42) to conclude

$$
\begin{aligned}
&= e^{-i\pi B}\sqrt{[d_B]}\sum_{X_5}\frac{v_V^{1/4}v_B^{1/4}v_{X_0}^{1/4}v_{X_1}^{1/4}}{v_A^{1/4}v_{X_5}^{1/2}}\left\{\begin{array}{cc|c} V & X_0 & X_5 \\ Y_0 & Z_1 & Z_0 \end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} B & V & A \\ X_0 & X_1 & X_5 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} B & X_5 & X_1 \\ Z_1 & Y_1 & Y_0 \end{array}\right\}_{(3)} \\
&\quad\times\sum_{Z_3}\frac{v_T^{1/4}v_V^{1/4}v_{Z_3}}{v_U^{1/2}v_W^{1/2}v_{Z_1}}\left\{\begin{array}{cc|c} U & W & T \\ Z_0 & Z_1 & Z_3 \end{array}\right\}_{(1)}\left\{\begin{array}{cc|c} W & U & V \\ Z_1 & Z_0 & Z_3 \end{array}\right\}_{(1)}
\end{aligned}
$$

= the righthandside of (61).

This concludes the proof that the expression of the reduced elements defines an intertwiner operator.

Let us now prove the final part of the theorem. Let us fix C such that $(C-|m_Z|)\in\mathbb{N}$ and assume that $\forall A,B,\ \left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & C \\ A & B & Z_0Z_1\end{array}\right]=0$. In this event, by multiplying the reduced element by $\left\{\begin{array}{cc|c} C & B & A \\ X_1 & X_0 & X_3\end{array}\right\}_{(1)}$, and summing over A, we would obtain, after the use of the orthogonality relation on $6j(1)$, $\forall B,\forall X_2,\ \left\{\begin{array}{cc|c} C & Z_1 & Z_0 \\ Y_0 & X_0 & X_2\end{array}\right\}_{(3)}\left\{\begin{array}{cc|c} B & Y_1 & Y_0 \\ Z_1 & X_2 & X_1\end{array}\right\}_{(3)}=0$. But from the orthogonality relation on $6j(3)$, $\exists X_2\ /\ \left\{\begin{array}{cc|c} C & Z_1 & Z_0 \\ Y_0 & X_0 & X_2\end{array}\right\}_{(3)}\neq 0$. As a result we obtain that $\left\{\begin{array}{cc|c} B & Y_1 & Y_0 \\ Z_1 & X_2 & X_1\end{array}\right\}_{(3)}=0$ for all B subject to the selection rules. From the behaviour of $\left\{\begin{array}{cc|c} B & Y_1 & Y_0 \\ Z_1 & X_2 & X_1\end{array}\right\}_{(3)}$ when B is large (see Eq. (103)), we obtain a contradiction. As a result the statement of the theorem holds true and implies in particular that the intertwiner is non zero. $\square$

4 Alternative Construction of Intertwiners in terms of the Quantum Haar Measure

Let us define the linear map $\hat{\Phi}[X_0X_1,Y_0Y_1]:V(X_0X_1)\otimes V(Y_0Y_1)\to\int^{\oplus}d(Z_0Z_1)H(Z_0Z_1)$ where $\forall w\in V(X_0X_1)\otimes V(Y_0Y_1), \hat{\Phi}[X_0X_1,Y_0Y_1](w)$ is the family of functions defined by $\hat{\Phi}[X_0X_1,Y_0Y_1](w)(Z_0Z_1)=N^{(X_0X_1)\,(Y_0Y_1)}_{(Z_0Z_1)}\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}(w)$, and $N^{(X_0X_1)\,(Y_0Y_1)}_{(Z_0Z_1)}$ are complex numbers depending on X_0,X_1,Y_0,Y_1,Z_0,Z_1.

We want to find N such that $\hat{\Phi}[X_0X_1,Y_0Y_1]$ is an isometry. As explained in the introduction, this is a delicate problem which requires another description of the space of intertwiners, where this isometry property is a direct consequence of Plancherel theorem.

Theorem 2 *Let $l\in V(X_0X_1), l'\in V(Y_0Y_1), v''\in V(Z_0Z_1)$, the following operator is well defined, and is an intertwiner of $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ module:*

$$\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l,l';v'']:V(X_0X_1)\otimes V(Y_0Y_1)\to V(Z_0Z_1),$$
$$\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l,l';v'']=\sum_{IJK}\overset{(Z_0Z_1)}{\Pi}(u^K)\dagger\,|v''><l\otimes l'|\overset{(X_0X_1)}{\Pi}(u^I)\otimes\overset{(Y_0Y_1)}{\Pi}(u^J)\ h(u_Iu_Ju_K).\quad(64)$$

Its matrix elements satisfy the relation:

$$\sum_{IJK}\overset{(X_0X_1)}{\Pi}{}^{A',i'}_{A,i}(u^I)\overset{(Y_0Y_1)}{\Pi}{}^{B',j'}_{B,j}(u^J)\overset{(Z_0Z_1)}{\Pi}{}^{\dagger C,k}_{C',k'}(u^K)\ \ h(u_Iu_Ju_K)=$$
$$=\sum_v\left(\begin{array}{c|cc} k & A & B \\ C & i & j\end{array}\right)\mu^{C_v}_{k'}\left(\begin{array}{cc|c} i' & j' & C' \\ A' & B' & v\end{array}\right)\left[\begin{array}{cc|c} A' & B' & C' \\ X_0X_1 & Y_0Y_1 & Z_0Z_1 \\ A & B & C\end{array}\right]$$

with $$\left[\begin{array}{cc|c} A' & B' & C' \\ X_0X_1 & Y_0Y_1 & Z_0Z_1 \\ A & B & C\end{array}\right]=\sum_{KLMN}\frac{[d_N][d_K]}{[d_C][d_{C'}]}\Lambda^{KL}_{A'A}(X_0X_1)\Lambda^{KM}_{B'B}(Y_0Y_1)\Lambda^{KN}_{C'C}(Z_1Z_0)\times$$
$$\times\left\{\begin{array}{cc|c} K & A & L \\ B & N & C\end{array}\right\}_{(0)}\left\{\begin{array}{cc|c} A' & B' & C' \\ K & N & M\end{array}\right\}_{(0)}\left\{\begin{array}{cc|c} B & K & M \\ A' & N & L\end{array}\right\}_{(0)}.\quad(65)$$

This infinite series converges absolutely and uniformly in ρ_X,ρ_Y,ρ_Z. Its square is a continuous function of ρ_X,ρ_Y,ρ_Z.

Proof:
We will use the convention of summation of repeated up and low small indices.

$$\sum_{IJK} {}^{(X_0X_1)}\Pi^{A',i'}_{A,i}(u^I)\ {}^{(X_0X_1)}\Pi^{B',j'}_{B,j}(u^J)\ {}^{(X_0X_1)}\Pi^{\dagger C,k}_{C',k'}(u^K)\ h(u_Iu_Ju_K)\ =$$

$$= \sum_{KLK'MK''N} \Lambda^{KL}_{A'A}(X_0X_1) \begin{pmatrix} r & s & L \\ A' & K & t \end{pmatrix} \begin{pmatrix} t & K & A \\ L & u & i \end{pmatrix} \Lambda^{K'M}_{B'B}(Y_0Y_1) \begin{pmatrix} a & b & M \\ B' & K' & c \end{pmatrix} \begin{pmatrix} c & K' & B \\ M & d & j \end{pmatrix} \times$$
$$\times \Lambda^{K''N}_{C'C}(Z_1Z_0) \begin{pmatrix} n & C' & K'' \\ N & m & p \end{pmatrix} \begin{pmatrix} l & k & N \\ K'' & C & n \end{pmatrix} \overset{C'}{w}_{qk'} (\overset{C'}{w}{}^{-1})^{mo} h(\overset{A'}{k}{}^{i'}_{r} \otimes E^u_s\ \overset{B'}{k}{}^{j'}_{a} \otimes E^d_b\ \overset{K'}{k}{}^{q}_{o} \overset{C'}{\otimes} \overset{K''}{E}{}^{p}_{l})$$

$$= \sum_{KLMN} \frac{e^{2i\pi K}[d_K]}{[d_{C'}]} \Lambda^{KL}_{A'A}(X_0X_1)\Lambda^{KM}_{B'B}(Y_0Y_1)\Lambda^{KN}_{C'C}(Z_1Z_0) \begin{pmatrix} r & s & L \\ A' & K & t \end{pmatrix} \begin{pmatrix} t & K & A \\ L & u & i \end{pmatrix} \begin{pmatrix} a & p & M \\ B' & K & c \end{pmatrix}$$
$$\times \begin{pmatrix} c & K & B \\ M & s & j \end{pmatrix} \begin{pmatrix} n & C' & K \\ N & m & p \end{pmatrix} \begin{pmatrix} l & k & N \\ K & C & n \end{pmatrix} \begin{pmatrix} m & A' & B' \\ C' & r & a \end{pmatrix} (\overset{K}{\mu}{}^{-1})^u_l \begin{pmatrix} i' & j' & C' \\ A' & B' & v \end{pmatrix} \overset{C'}{\mu}{}^{v}_{k'}$$

after the use of formulae (57) and (8)

$$= \sum_{KLMN} \frac{e^{2i\pi K}[d_K]}{[d_{C'}]} \Lambda^{KL}_{A'A}(X_0X_1)\Lambda^{KM}_{B'B}(Y_0Y_1)\Lambda^{KN}_{C'C}(Z_1Z_0) \left\{ \begin{matrix} A' & B' & C' \\ K & N & M \end{matrix} \right\}_{(0)} \begin{pmatrix} r & s & L \\ A' & K & t \end{pmatrix}$$
$$\times \begin{pmatrix} t & K & A \\ L & u & i \end{pmatrix} \begin{pmatrix} n & A' & M \\ N & r & c \end{pmatrix} \begin{pmatrix} c & K & B \\ M & s & j \end{pmatrix} \begin{pmatrix} l & k & N \\ K & C & n \end{pmatrix} (\overset{K}{\mu}{}^{-1})^u_l \begin{pmatrix} i' & j' & C' \\ A' & B' & v \end{pmatrix} \overset{C'}{\mu}{}^{v}_{k'}$$

$$= \sum_{KLMN} \frac{e^{2i\pi K}[d_K]}{[d_{C'}]} \Lambda^{KL}_{A'A}(X_0X_1)\Lambda^{KM}_{B'B}(Y_0Y_1)\Lambda^{KN}_{C'C}(Z_1Z_0) \left\{ \begin{matrix} A' & B' & C' \\ K & N & M \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} B & K & M \\ A' & N & L \end{matrix} \right\}_{(0)}$$
$$\times \begin{pmatrix} t & K & A \\ L & u & i \end{pmatrix} \begin{pmatrix} n & L & B \\ N & t & j \end{pmatrix} \begin{pmatrix} l & k & N \\ K & C & n \end{pmatrix} (\overset{K}{\mu}{}^{-1})^u_l \begin{pmatrix} i' & j' & C' \\ A' & B' & v \end{pmatrix} \overset{C'}{\mu}{}^{v}_{k'}$$

after having used twice the formula (11)

$$= \sum_{KLMN} \frac{e^{2i\pi K}[d_K]}{[d_{C'}]} \Lambda^{KL}_{A'A}(X_0X_1)\Lambda^{KM}_{B'B}(Y_0Y_1)\Lambda^{KN}_{C'C}(Z_1Z_0) \left\{ \begin{matrix} A' & B' & C' \\ K & N & M \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} B & K & M \\ A' & N & L \end{matrix} \right\}_{(0)}$$
$$\times \frac{e^{2i\pi(N-C)}[d_N]}{[d_C]} \begin{pmatrix} l & t & A \\ K & L & i \end{pmatrix} \begin{pmatrix} n & L & B \\ N & t & j \end{pmatrix} \begin{pmatrix} k & K & N \\ C & l & n \end{pmatrix} (\overset{K}{\mu}{}^{-1})^u_l \begin{pmatrix} i' & j' & C' \\ A' & B' & v \end{pmatrix} \overset{C'}{\mu}{}^{v}_{k'}$$
$$= \text{righthandside of (65)},$$

where we have used formulae (7) and (11) to conclude.

This proof holds true as soon as we have shown that these series are absolutely convergent. Proof of the convergence of the last series can be obtained using asymptotic properties of coefficients Λ^{BC}_{AD}, as well as asymptotic properties of $6j(0)$. Indeed, using the selection rules of $6j(0)$, we know that in the four sums

$$\sum_{Klmn} \frac{[d_{K+n}][d_K]}{[d_C][d_{C'}]} \Lambda^{K\,K+l}_{A'A}(X_0X_1)\Lambda^{K\,K+m}_{B'B}(Y_0Y_1)\Lambda^{K\,K+n}_{C'C}(Z_1Z_0) \times$$
$$\times \left\{ \begin{matrix} K & A & K+l \\ B & K+n & C \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} A' & B' & C' \\ K & K+n & K+m \end{matrix} \right\}_{(0)} \left\{ \begin{matrix} B & K & K+m \\ A' & K+n & K+l \end{matrix} \right\}_{(0)}, \quad (66)$$

for a fixed K, the range of the sum over l, m, n is finite and fixed by $|l| \leq \min(A, A'), |m| \leq \min(B, B'), |n| \leq \min(C, C'), |l-n| \leq B, |n-m| \leq A'$. Thus, in order to show the absolute convergence of this series, it is sufficient to bound the general term in K by a geometric series. This can be proved using the behaviour of the coefficients Λ^{KK+l}_{AD} when K is large, derived in the appendix. Precisely, using (111) and (16)(17), we have the following property: $\forall l, m, n, A, B, C, A', B', C' \in \frac{1}{2}\mathbb{Z}$, there exist $Q, Q' \in \mathbb{R}$, such that $\forall \rho_X, \rho_Y, \rho_Z \in]-\frac{\pi}{\hbar}, \frac{\pi}{\hbar}]$:

$$\forall K, \left| \frac{[d_{K+n}][d_K]}{[d_C][d_{C'}]} \Lambda^{K\,K+l}_{A'A}(X_0X_1) \Lambda^{K\,K+m}_{B'B}(Y_0Y_1) \Lambda^{K\,K+n}_{C'C}(Z_1Z_0) \right| \leq QK^3 q^{2K}$$

$$\forall K, \left| \left\{ \begin{array}{cc|c} K & A & K+l \\ B & K+n & C \end{array} \right\}_{(0)} \left\{ \begin{array}{cc|c} A' & B' & C' \\ K & K+n & K+m \end{array} \right\}_{(0)} \left\{ \begin{array}{cc|c} B & K & K+m \\ A' & K+n & K+l \end{array} \right\}_{(0)} \right| \leq Q'. \quad (67)$$

The series is therefore absolutely convergent and uniformly convergent in ρ_X, ρ_Y, ρ_Z. From this last result and using the property that $\mathcal{N}^{(A)}(q^{2X_1+1}, m_X)\mathcal{N}^{(D)}(q^{2X_1+1}, m_X)^{-1}\Lambda^{BC}_{AD}(X_0X_1)$ is a continuous functions of ρ_X, the square of the matrix coefficients of the operator $\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ are continuous functions of the three variables ρ_X, ρ_Y, ρ_Z. Note that in the case where $A = A', B = B', C = C'$ the expression (65) is itself a continuous function of ρ_X, ρ_Y, ρ_Z.

We still have to show that the linear map $\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v'']$ is an intertwiner. We have

$$\begin{aligned} &\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v''] = \\ &= \sum_{IJK} \overset{(Z_0Z_1)}{\Pi}(u^K) \dagger |v'' >< l \otimes l'| \overset{(X_0X_1)}{\Pi}(u^I) \otimes \overset{(Y_0Y_1)}{\Pi}(u^J)\, h(u_I u_J u_K) \\ &= \sum_{IJK} \overset{(Z_0Z_1)}{\Pi}(S^{-1}(u^K))|v'' >< l \otimes l'| \overset{(X_0X_1)}{\Pi}(u^I) \otimes \overset{(Y_0Y_1)}{\Pi}(u^J)\, h(u_I u_J u_K) \end{aligned}$$

Using the identity verified by any Haar measure on a co-quasitriangular Hopf algebra A [8, 9]

$$h(ab) = \sum_{(a)} h(ba_{(2)}) < \mu, a_{(1)} >< \mu, a_{(3)} >, \quad \forall a, b \in A, \quad (68)$$

and simple transformations using properties of the antipode and of the group-like element μ, we obtain that $\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v'']$ is related to the operator:

$$\begin{aligned} &\Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v''] : V(X_0X_1) \otimes V(Y_0Y_1) \to V(Z_0Z_1), \quad (69) \\ &\Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v''] = \sum_{IJK} \overset{(Z_0Z_1)}{\Pi}(S(u^K)\mu)|v'' >< l \otimes l'| \overset{(X_0X_1)}{\Pi}(u^I) \otimes \overset{(Y_0Y_1)}{\Pi}(u^J)\, h(u_K u_I u_J), \end{aligned}$$

as follows:

$\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v''] = \sum_{C,r} \Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; \overset{C}{e}_r] < \overset{CC}{\mu} \overset{}{e}_r, v'' >$.

As a result it is equivalent to show that $\Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l, l'; v'']$ is an intertwiner operator. This is a simple consequence of the absolute convergence of the series and of the right invariance of the Haar measure. $\square$

The following proposition gives the link between intertwiners constructed using the Haar measure and the reduced elements.

Proposition 11 *There exists a real number $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ such that*

$$\left[\begin{array}{cc|c} A' & B' & C' \\ X_0X_1 & Y_0Y_1 & Z_0Z_1 \\ A & B & C \end{array} \right] = \mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)} \left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & C \\ A & B & Z_0Z_1 \end{array} \right] \overline{\left[\begin{array}{cc|c} X_0X_1 & Y_0Y_1 & C' \\ A' & B' & Z_0Z_1 \end{array} \right]}. \quad (70)$$

Proof:

The operator $\Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l,l';v'']$ being an intertwiner,the theorem 1 ensures the existence of a complex number $\lambda(l,l';v'')$ such that: $\Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l,l';v''] = \lambda(l,l';v'')\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$. We will denote $\Pi = \overset{(X_0X_1)}{\Pi}, \Pi' = \overset{(Y_0Y_1)}{\Pi}, \Pi'' = \overset{(Z_0Z_1)}{\Pi}$ and we have the following sequence of equalities:

$$\overline{\lambda(l,l';v'')}\ \ \overline{< l''|\Phi^{\Pi\,\Pi'}_{\Pi''}|v\otimes v'>} =$$
$$= \sum_{IJK} < v''|\Pi''((S(e^I)\mu)^*)|l''>< v\otimes v'|\Pi(e^{J*})\otimes\Pi'(e^{K*})|l\otimes l'> \overline{h(e_Ie_Je_K)} =$$
$$= \sum_{IJK} < v''|\Pi''(\mu S^{-1}(e^{I*}))|l''>< v\otimes v'|\Pi(e^{J*})\otimes\Pi'(e^{K*})|l\otimes l'> h(e^*_Ke^*_Je^*_I) =$$
$$= \sum_{IJK} < v''|\Pi''(\mu S^{-1}(e^{I}))|l''>< v\otimes v'|\Pi(e^{J})\otimes\Pi'(e^{K})|l\otimes l'> h(S^{-1}(e_K)S^{-1}(e_J)S^{-1}(e_I)) =$$
$$= \sum_{IJK} < v''|\Pi''(S(e^I)\mu)|l''>< v\otimes v'|\Pi(e^{J})\otimes\Pi'(e^{K})|l\otimes l'> h(e_Ie_Je_K) =$$
$$= \lambda(v,v';l'')\ \ < v''|\Phi^{\Pi\,\Pi'}_{\Pi''}|l\otimes l'> .$$

As a result

$$\lambda(v,v';l'') = \mathcal{M}^{\Pi\,\Pi'}_{\Pi''}\ \overline{< l''|\Phi^{\Pi\,\Pi'}_{\Pi''}|v\otimes v'>},$$

from which we get

$$\sum_{IJK} < l''|\Pi''(S(e^I)\mu)|v''>< l|\Pi(e^J)|v>< l'|\Pi'(e^K)|v'>\ \ h(e_Ie_Je_K) =$$
$$= \mathcal{M}^{\Pi\,\Pi'}_{\Pi''} < l''|\Phi^{\Pi\,\Pi'}_{\Pi''}|v\otimes v'> \overline{< v''|\Phi^{\Pi\,\Pi'}_{\Pi''}|l\otimes l'>}. \tag{71}$$

Using the relation between the operator $\Xi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l,l';v'']$ and $\Upsilon^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}[l,l';v'']$ we obtain the announced result. $\Box$

In order to make a precise connection, later on, between matrix elements of $\hat{\Phi}[X_0X_1,Y_0Y_1]$ and Askey-Wilson polynomials we need an explicit formula for $\mathcal{M}$.

Proposition 12 *The normalization factor $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ is a positive number and is given by the following expression:*

$$\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)} = |\frac{q^{2(X_0-X_1+Y_0-Y_1+Z_0-Z_1)}(1-q^2)(1)^2_\infty}{\nu_1(2X_0+1)^2\nu_1(2Y_0+1)^2\nu_1(2Z_0+1)^2}| \times$$
$$\times\frac{|\xi(2X_1+1)\xi(2Y_1+1)\xi(2Z_1+1)|}{|\xi(X_1+Y_1-Z_1+1)\xi(X_1-Y_1+Z_1+1)\xi(Y_1+Z_1-X_1+1)\xi(X_1+Y_1+Z_1+2)|} \tag{72}$$

where we have defined the function ξ, by $\xi(z) = (z)_\infty(1-z)_\infty$.

Proof:

We will identify certain asymptotics of the lefthandside and the righthandside of (70) in order to compute the normalization $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$. Let us first find the behaviour of

$$f(T) = \left[\begin{array}{cc|c} P{+}T & P & T\\ X_0X_1 & Y_0Y_1 & Z_0Z_1\\ P{+}T & P & T\end{array}\right] \quad \text{when } T\to+\infty,\ \ P = Y_0-Y_1>0. \tag{73}$$

It can be seen from the properties (16)(17)(113), that the unique leading term in the explicit expression :

$$f(T)=\sum_{KlMn}\frac{[d_{T+n}][d_K]}{[d_T]^2}\Lambda^{K\ P+T+l}_{P+T}(X_0X_1)\Lambda^{K\ M}_{\ P}(Y_0Y_1)\Lambda^{K\ T+n}_{\ T}(Z_1Z_0)\times$$
$$\times\left\{\begin{matrix}K & P{+}T\\ P & T{+}n\end{matrix}\middle|\begin{matrix}P{+}T{+}l\\ T\end{matrix}\right\}_{(0)}\left\{\begin{matrix}K & P\\ P{+}T & T{+}n\end{matrix}\middle|\begin{matrix}M\\ T\end{matrix}\right\}_{(0)}\left\{\begin{matrix}P & K\\ P{+}T & T{+}n\end{matrix}\middle|\begin{matrix}M\\ P{+}T{+}l\end{matrix}\right\}_{(0)},$$

is the term corresponding to $l=n=0$. The behaviour when $T\to+\infty$ of $f(T)$ is therefore given by

$$f(T)\sim q^{2T}(1-q^2)\sum_{KM}[d_K]\left(\begin{matrix}-P\\ M\end{matrix}\middle|\begin{matrix}K & P\\ 0 & -P\end{matrix}\right)\left(\begin{matrix}-P & 0\\ P & K\end{matrix}\middle|\begin{matrix}M\\ -P\end{matrix}\right)\Lambda^{K\ M}_{\ P}(Y_0Y_1).$$

The series in the right handside can be exactly computed using the formula (114) proved in the appendix. We finally obtain, for $P=Y_0-Y_1>0$:

$$\left[\begin{matrix}P{+}T & P\\ X_0X_1 & Y_0Y_1\\ P{+}T & P\end{matrix}\middle|\begin{matrix}T\\ Z_0Z_1\\ T\end{matrix}\right]\sim_{T\to+\infty} q^{2T}(1-q^2)\frac{\hbar[d_{|Y_0-Y_1|}]}{2\pi\mathcal{P}(Y_0Y_1)}. \tag{74}$$

From the fact that the righthandside of the last expression is positive, and that the right handside of (70) is in this situation a modulus square we obtain that $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ is a positive number. Let us now compute the behaviour of the righthandside of (70) for $T\to+\infty, P=m_Y$ (this computation has been done for $0\leq m_Y$, but for $m_Y<0$ a similar proof can be done). In the appendix, we have proved the following result:

$$\left[\begin{matrix}X_0X_1 & Y_0Y_1\\ T{+}Y_0{-}Y_1 & Y_0{-}Y_1\end{matrix}\middle|\begin{matrix}T\\ Z_0Z_1\end{matrix}\right]\sim e^{-i\pi 2T}q^{-T(Y_0+Y_1)}\ e^{i\pi(\frac{X_0-X_1}{2})}q^{-Z_0Z_1}\sqrt{[d_{Y_0-Y_1}]}\times$$
$$\times\nu_1(2X_1{+}1)\nu_1(2Z_0{+}1)\frac{\varphi(2X_0,P{+}T{+}X_0{-}X_1)\varphi(Z_1{+}X_1{-}Y_1,Z_1{-}Z_0{+}X_1{-}X_0{+}Y_0{-}Y_1)}{\varphi(Z_0{+}Z_1,T)}\times$$
$$\times\frac{\nu_\infty(Y_0{+}X_0{-}Z_0{+}1)\nu_\infty(Y_0{+}X_0{+}Z_0{+}2)\nu_\infty(Y_0{-}X_0{-}Z_0)\nu_\infty(Y_0{+}Z_0{-}X_0{+}1)}{q^{\frac{3}{2}X_0^2+\frac{1}{2}X_0-\frac{1}{2}Z_0^2+\frac{1}{2}Z_0}(1)_\infty\nu_\infty(2Y_1{+}1)\nu_\infty(2Y_0{+}1)}\times$$
$$\times\frac{\nu_\infty(Y_1{+}X_1{-}Z_1{+}1)\nu_\infty(Y_1{+}X_1{+}Z_1{+}2)\nu_\infty(Y_1{-}X_1{-}Z_1)\nu_\infty(Y_1{+}Z_1{-}X_1{+}1)}{q^{-\frac{3}{2}X_1^2-\frac{1}{2}X_1-\frac{1}{2}Z_1^2-\frac{1}{2}Z_1}\nu_\infty(2X_0{+}1)\nu_\infty(-2X_0)\nu_\infty(Z_0+Z_1+1)\nu_\infty(-Z_0{-}Z_1)}. \tag{75}$$

From (70), and after elementary algebraic relations on q-factorials, we finally get the announced expression for $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$. $\square$

The formula (70) is an identity where the left hansdside is a complicated series, and the right handside is the infinite product ($\mathcal{M}$), multiplied by square roots of rational fractions. Let us give an example of this formula in its simplest case, which is achieved when $A=A'=B=B'=C=C'=0$. We necessarily have $m_X=m_Y=m_Z=0$, i.e $X_0=X_1, Y_0=Y_1, Z_0=Z_1$. In this case we can compute $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ directly. Indeed:

$$\left[\begin{matrix}0 & 0\\ X_0X_1 & Y_0Y_1\\ 0 & 0\end{matrix}\middle|\begin{matrix}0\\ Z_0Z_1\\ 0\end{matrix}\right]=\sum_B[d_B]^2\Lambda^{BB}_0(X_0X_1)\Lambda^{BB}_0(Y_0Y_1)\Lambda^{BB}_0(Z_0Z_1). \tag{76}$$

Using the identity $\Lambda^{BB}_0(X_0X_0)=\frac{[(2X_0+1)(2B+1)]}{[2X_0+1][2B+1]}$, the formula (Exercice 12 of [22])

$$4\sum_{n=1}^{+\infty}q^n\frac{sin(2nz)}{1-q^{2n}}=\frac{\theta_4'(z,\tau)}{\theta_4(z,\tau)} \tag{77}$$

where $q = e^{-\hbar} = e^{i\pi\tau}$, and the Jacobi's fundamental formulae ([22] 21.22):

$$2\theta_4(x')\theta_4(y')\theta_4(z')\theta_4(w') = \sum_{j=1}^{4}(-1)^{\delta_{j,2}}\theta_j(x)\theta_j(y)\theta_j(z)\theta_j(w) \tag{78}$$

where $2w' = -w + x + y + z$ (and circular permutation to define x',y',z'), we obtain that the series (76) is equal to:

$$\frac{(1-q^2)q^{-3/4}(1)^3_\infty\theta_1(2\lambda_x)\theta_1(2\lambda_y)\theta_1(2\lambda_z)(sin(2\lambda_x)sin(2\lambda_y)sin(2\lambda_z))^{-1}}{8\theta_4(\lambda_x+\lambda_y+\lambda_z)\theta_4(-\lambda_x+\lambda_y+\lambda_z)\theta_4(\lambda_x-\lambda_y+\lambda_z)\theta_4(\lambda_x+\lambda_y-\lambda_z)} \tag{79}$$

where we have chosen $2X_0+1 = i\frac{4\lambda_x}{\hbar}, 2Y_0+1 = i\frac{4\lambda_y}{\hbar}, 2Z_0+1 = i\frac{4\lambda_z}{\hbar}$, and $\theta_j, (j = 1, ..4)$ are the usual Theta functions, as defined in [22]. Note that this expression is exactly the coefficient $\mathcal{M}^{(X_0X_0)(Y_0Y_0)}_{(Z_0Z_0)}$, once we have expressed the Theta functions in term of infinite products. However such a simple derivation of $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$, does not generalize so easily to the case where m_X, m_Y, m_Z are non zero.
Formula (79) is the quantum deformation of a formula first found in [12].

Remark: We will need in the next section the following identity, which can be easily proved using (72). $\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$, can also be written as:

$$\begin{aligned}\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)} &= \psi_1 \times \frac{q^{Z_0-3Z_1+Y_0-3Y_1+X_0-3X_1-3}(1)^2_\infty(1-q^2)}{\nu_1(2X_0+1)\nu_1(-2X_1-1)\nu_1(2Y_0+1)\nu_1(-2Y_1-1)\nu_1(2Z_0+1)\nu_1(-2Z_1-1)} \times \\ &\times \frac{\xi(2X_1+1, 2Y_1+1, 2Z_1+1)}{\xi(Y_1+X_1-Z_1+1, X_1+Z_1-Y_1+1, Y_1+Z_1-X_1+1, X_1+Y_1+Z_1+2)}\end{aligned} \tag{80}$$

where we have denoted $\xi(a_1, \cdots, a_n) = \prod_{k=1}^n \xi(a_k)$ and where ψ_1 is the fourth root of unit given by:

$$\psi_1 = \frac{\varphi_{(2Y_1, 2Y_1-2Y_0-1)} \times (\text{cycl. perm. on } X, Y, Z))}{\varphi_{(Y_1+X_1-Z_1+1, Y_1-Y_0+X_1-X_0-Z_0+Z_1)} \times (\text{cycl. perm. on } X, Y, Z) \times \varphi_{(Y_1+X_1+Z_1+2, Y_1-Y_0+X_1-X_0+Z_1-Z_0)}}$$

5 Decomposition Theorem of the Tensor Product of Principal Representations and Askey-Wilson Polynomials

5.1 Decomposition Theorem of the Tensor Product of Principal Representations

Let us now prove that, chosing $N^{(X_0X_1)\,(Y_0Y_1)}_{(Z_0Z_1)} = (\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)})^{1/2} > 0$, $\hat{\Phi}[X_0X_1, Y_0Y_1]$ is an isometry.

Theorem 3 $\hat{\Phi}[X_0X_1, Y_0Y_1] : V(X_0X_1) \otimes V(Y_0Y_1) \to \int^{\oplus} d(Z_0Z_1)H(Z_0Z_1)$ *is an isometry. This is a direct consequence of the following identity:*

$$\forall C \in \frac{1}{2}\mathbb{Z}^+, \int d\mathcal{P}(Z_0Z_1)\left[\begin{array}{cc|c} A' & B' & C \\ X_0X_1 & Y_0Y_1 & Z_0Z_1 \\ A & B & C \end{array}\right] = \delta_{A,A'}\delta_{B,B'}Y^{(1)}_{(A,X_0-X_1)}Y^{(1)}_{(B,Y_0-Y_1)}. \tag{81}$$

Proof:

$V(X_0X_1)\otimes V(Y_0Y_1)$ is a $\mathfrak{U}_q(su(2))$ module, let us denote by $(V(X_0X_1)\otimes V(Y_0Y_1))^{[C]}$ its isotypic component of spin C. From the definition of $\hat{\Phi}[X_0X_1,Y_0Y_1]$ and the choice $N=(\mathcal{M}^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)})^{1/2}$, it is simple to show that the restriction of $\hat{\Phi}[X_0X_1,Y_0Y_1]$ to $(V(X_0X_1)\otimes V(Y_0Y_1))^{[C]}$ is an isometry if and only if (81) is satisfied for this C. We already proved that $\left[\begin{array}{cc|c} A' & B' & C \\ X_0X_1 & Y_0Y_1 & Z_0Z_1 \\ A & B & C \end{array}\right]$ is a piecewise continuous function of ρ_Z, as a result it is integrable, and using formula (65) and the uniform convergence properties of this series, we can invert the integral and the sum:

$$\begin{aligned}
&\text{lefthandside of (81)} = \\
&= \sum_{KLMN} \frac{[d_N][d_K]}{[d_C]^2} \Lambda^{KL}_{A'A}(X_0X_1)\Lambda^{KM}_{B'B}(Y_0Y_1)\int d\mathcal{P}(Z_0Z_1)\Lambda^{KN}_{A'A}(Z_1Z_0)\times \\
&\qquad\qquad \times \left\{\begin{array}{cc|c} K & A & L \\ B & N & C \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} A' & B' & C \\ K & N & M \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} B & K & M \\ A' & N & L \end{array}\right\}_{(0)}
\end{aligned}$$

Using the Plancherel formula (59), we obtain that this series is equal to:

$$\begin{aligned}
&= \delta_{A,A'}\delta_{B,B'}\Lambda^{0A}_{AA}(X_0X_1)\Lambda^{0B}_{BB}(Y_0Y_1) \left\{\begin{array}{cc|c} 0 & A & A \\ B & C & C \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} A & B & C \\ 0 & C & B \end{array}\right\}_{(0)} \left\{\begin{array}{cc|c} B & 0 & B \\ A & C & A \end{array}\right\}_{(0)} = \\
&= \text{righthandside of (81)}.
\end{aligned}$$

□

$\hat{\Phi}[X_0X_1,Y_0Y_1]$ being an isometry from $V(X_0X_1)\otimes V(Y_0Y_1)$ to the Hilbert space $\int^{\oplus} d(Z_0Z_1)H(Z_0Z_1)$, there exists a unique isometry $\Phi^{\#}[X_0X_1,Y_0Y_1]$ from the tensor product of Hilbert space $H(X_0X_1)\otimes H(Y_0Y_1)$ to $\int^{\oplus} d(Z_0Z_1)H(Z_0Z_1)$ which restriction to $V(X_0X_1)\otimes V(Y_0Y_1)$ gives back $\hat{\Phi}[X_0X_1,Y_0Y_1]$.

Theorem 4 $\Phi^{\#}[X_0X_1,Y_0Y_1] : H(X_0X_1)\otimes H(Y_0Y_1) \to \int^{\oplus} d(Z_0Z_1)H(Z_0Z_1)$ *is an invertible isometry of Hilbert space.*

Proof:

It is trivial that $\Phi^{\#}[X_0X_1,Y_0Y_1]$ is an isometry, as a result it is injective. We therefore have only to show that $\Phi^{\#}[X_0X_1,Y_0Y_1]$ is surjective. The proof goes along the same lines as the proof of the surjectivity of the Fourier transform in [1]. It is sufficient to show that $E=\Phi^{\#}[X_0X_1,Y_0Y_1](V(X_0X_1)\otimes V(Y_0Y_1))$ is dense in $\int^{\oplus} d(Z_0Z_1)H(Z_0Z_1)$, which is equivalent to show that $E^{\perp}=\{0\}$. Let $f\in\int^{\oplus} d(Z_0Z_1)H(Z_0Z_1)$, and assume that $< f, \Phi[X_0X_1,Y_0Y_1](\overset{A}{e}_i (X_0X_1)\otimes \overset{B}{e}_j (Y_0Y_1)) >= 0, \forall A,B,i,j$. This in particular implies that

$$< f, \Phi[X_0X_1,Y_0Y_1](\Delta(a)(\overset{A}{e}_i (X_0X_1)\otimes \overset{B}{e}_j (Y_0Y_1))) >= 0 \qquad (82)$$

for any element a in the center of $\mathfrak{U}_q(sl(2,\mathbb{C}))_{\mathbb{R}}$. As a result, by taking $a=\Omega_+^p\Omega_-^r$ and using the property that $\Phi^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}$ is an intertwiner, we get the identity: $\forall p,r\in\mathbb{N}, \forall A,B,C\in\frac{1}{2}\mathbb{Z}^+, \forall k=-C,..,C$

$$\int d(Z_0Z_1) < f| \overset{C}{e}_k (Z_0Z_1) > \omega_+(Z_0Z_1)^p\omega_-(Z_0Z_1)^r g(X_0X_1,Y_0,Y_1,Z_0,Z_1) = 0, \qquad (83)$$

where $g(X_0X_1, Y_0, Y_1, Z_0, Z_1) = \left[\begin{matrix} X_0X_1 & Y_0Y_1 \\ A & B \end{matrix}\middle|\begin{matrix} C \\ Z_0Z_1 \end{matrix}\right] N^{(X_0X_1)\,(Y_0Y_1)}_{(Z_0Z_1)}$. $|g|^2$ being a continuous function in ρ_Z, $< f|\overset{C}{e}_k> g$ is an L^2 function in the variable ρ_Z. We can now apply the argument of the proof of Plancherel Formula in [1]and conclude that the set of equations (83) for all p, r implies that $< f|\overset{C}{e}_k> g = 0$ as a L^2 function. From theorem 1, we can always find for every C, a couple A, B such that $g \neq 0$. From the explicit form of g we know that the zeroes of g considered as a ρ_Z function are finite, we obtain that $< f|\overset{C}{e}_k>= 0$, which implies $f = 0$. □

5.2 R-matrix in the Tensor Product of Infinite dimensional Representations

In this section we will compute the R-matrix in the tensor product of two irreducible infinite dimensional representations $(\overset{(X_0X_1)}{\Pi} \otimes \overset{(Y_0Y_1)}{\Pi})$. We will in particular show that the expression of $(\overset{(X_0X_1)}{\Pi} \otimes \overset{(Y_0Y_1)}{\Pi})(\mathcal{R})$ is an operator which domain contains all the vectors of the form $\overset{A}{e}_m(X_0X_1)\otimes \overset{B}{e}_n(Y_0Y_1)$, and more precisely we have the following proposition:

Proposition 13 *The expression of the R matrix of $\mathfrak{U}_q(sl(2,\mathbb{C})_\mathbb{R})$ represented on $\overset{(X_0X_1)}{V} \otimes \overset{(Y_0Y_1)}{V}$ is given by the following action:*

$$(\overset{(X_0X_1)}{\Pi} \otimes \overset{(Y_0Y_1)}{\Pi})(\mathcal{R})(\overset{B}{e}_m \otimes \overset{C}{e}_n) = \sum_{DF} \overset{B}{e}_j \otimes \overset{F}{e}_p \left(\begin{matrix} p & j \\ F & B \end{matrix}\middle|\begin{matrix} D \\ x \end{matrix}\right)\left(\begin{matrix} x \\ D \end{matrix}\middle|\begin{matrix} B & C \\ m & n \end{matrix}\right) \Lambda^{BD}_{FC}(X_0, X_1). \quad (84)$$

In particular this expression is a finite sum although the universal formula for $\mathcal{R}$ is an infinite sum.

Proof:

The element $\overset{A}{L}$ are multipliers of $\tilde{\mathfrak{U}}_q(sl(2,\mathbb{C})_\mathbb{R})$ and we have the trivial relation $\overset{A}{L}{}^{(\pm)i}_{\ j} = \sum_B^{\oplus} \overset{AB}{R}{}^{ik}_{jl} \overset{B}{X}{}^l_k$. The R matrix of $\mathfrak{U}_q(sl(2,\mathbb{C})_\mathbb{R})$ is written as: $\mathcal{R} = \sum_A \overset{A}{X}{}^i_j \otimes 1 \otimes 1 \otimes \overset{A}{g}{}^j_i$. From the expression of a representation $\tilde{\Pi}$ of $\tilde{\mathfrak{U}}_q(sl(2,\mathbb{C})_\mathbb{R})$, associated to (X_0, X_1) a straightforward computation of the representation of the R-matrix gives the formula of the statement. □

Remark. This situation is in sharp contrast with the case of $SU_q(1,1)$ for q real. The structure of Hopf algebra on $\mathfrak{U}_q(su(1,1))$ is the same as $\mathfrak{U}_q(su(2))$, the only difference is in the definition of the star structure: $J_z^\star = J_z, J_+^\star = -J_-, J_-^\star = -J_+$. It is easy to classify the irreducible unitary representations, the principal unitary representation of $\mathfrak{U}(su(1,1))$ being now easily quantized as follows:

$$J_z.e_m = me_m \qquad J_+.e_m = [m-\tau+\epsilon]e_{m+1} \qquad J_-.e_m = -[m+\tau+\epsilon]e_{m+1}$$

where as usual $\epsilon \in \{0, \frac{1}{2}\}, \tau = i\nu - \frac{1}{2}, \nu \in \mathbb{R}$, and e_m is an orthonormal basis of the representation. The universal R matrix is still formally defined by $R = q^{2J_z\otimes J_z} e_{q^{-1}}^{(q-q^{-1})\ (q^{J_z}J_+\otimes J_- q^{-J_z})}$. It is easy to see that $(\pi\otimes\pi')(R)(e_m\otimes e'_n)$ has no meaning in the l^2 sense except in the trivial case where $q = 1$. Indeed a straightforward computation shows that a formal expansion of $(\pi\otimes\pi')(R)(e_m\otimes e'_n)$ gives $(\pi\otimes\pi')(R)(e_m\otimes e'_n) = \sum_{p=0}^{+\infty} a_p(e_{m+p}\otimes e'_{n-p})$ with $a_p \sim (q-q^{-1})\gamma q^{-2p^2+\alpha p+\beta}$ for $0 < q < 1$ with $\gamma \neq 0$.

5.3 Connection with Askey-Wilson Polynomials.

In the sequel, we will show that we can reexpress $N^{(X_0X_1)\,(Y_0Y_1)}_{(Z_0Z_1)}\left[\begin{matrix}X_0X_1 & Y_0Y_1\\ A & B\end{matrix}\middle|\begin{matrix}C\\ Z_0Z_1\end{matrix}\right]$ in terms of q-Racah polynomials and Askey-Wilson polynomials, and that the relation (81) is an orthogonality property mixing q-Racah polynomials and Askey-Wilson polynomials in a non trivial way.

Lemma 3 *The following three identities are satisfied:*

$$1)\quad \overline{\left[\begin{matrix}X_0X_1 & Y_0Y_1\\ A & B\end{matrix}\middle|\begin{matrix}C\\ Z_0Z_1\end{matrix}\right]} = \psi_2\left[\begin{matrix}\underline{X_1X_0} & \underline{Y_1Y_0}\\ A & B\end{matrix}\middle|\begin{matrix}C\\ \underline{Z_1Z_0}\end{matrix}\right]$$

$$2)\quad \left[\begin{matrix}\underline{X_1X_0} & \underline{Y_1Y_0}\\ A & B\end{matrix}\middle|\begin{matrix}C\\ \underline{Z_1Z_0}\end{matrix}\right] = \psi_3\left[\begin{matrix}X_1X_0 & Y_1Y_0\\ A & B\end{matrix}\middle|\begin{matrix}C\\ Z_1Z_0\end{matrix}\right]$$

$$3)\quad \left[\begin{matrix}X_1X_0 & Y_1Y_0\\ A & B\end{matrix}\middle|\begin{matrix}C\\ Z_1Z_0\end{matrix}\right] = e^{-i\pi B}\sqrt{[d_B]}\sum_{X_2}\left\{\begin{matrix}T & Z_1\\ Y_0 & X_0\end{matrix}\middle|\begin{matrix}Z_0\\ X_2\end{matrix}\right\}_{(3)}\left\{\begin{matrix}C & B\\ X_1 & X_0\end{matrix}\middle|\begin{matrix}A\\ X_2\end{matrix}\right\}_{(1)}\times$$
$$\times\left\{\begin{matrix}B & Y_1\\ Z_1 & X_2\end{matrix}\middle|\begin{matrix}Y_0\\ X_1\end{matrix}\right\}_{(3)}\frac{v_A^{1/4}v_{X_2}^{1/2}}{v_B^{1/4}v_C^{1/4}v_{X_0}^{1/4}v_{X_1}^{1/4}} \tag{85}$$

where $\psi_2 = e^{i\pi(B+C+m_X+A)}$*, and* ψ_3 *is such that*

$$\psi_1\psi_2\psi_3 = e^{i\pi(m_X+m_Y-m_Z)}\varphi_{(2X_1+1,-1)}\varphi_{(2Y_1+1,-1)}\varphi_{(2Z_1+1,-1)}$$

Proof:

The identity 1) follows from a very simple computation. The identity 2) follows from the use of propositions (2) and (9). Identity 3) is a direct consequence of a Racah and a pentagonal equation on $6j(3)$ symbols. □

Using these identities theorem (3) can be reformulated as follows

Proposition 14

$$\int d(Z_0Z_1)\Gamma^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}\left\{\begin{matrix}A & Z_1\\ Y_0 & X_0\end{matrix}\middle|\begin{matrix}Z_0\\ X_2\end{matrix}\right\}_{(3)}\left\{\begin{matrix}A & Z_1\\ Y_0 & X_0\end{matrix}\middle|\begin{matrix}Z_0\\ X_3\end{matrix}\right\}_{(3)}\left\{\begin{matrix}B & Y_1\\ Z_1 & X_2\end{matrix}\middle|\begin{matrix}Y_0\\ X_1\end{matrix}\right\}_{(3)}\left\{\begin{matrix}C & Y_1\\ Z_1 & X_3\end{matrix}\middle|\begin{matrix}Y_0\\ X_1\end{matrix}\right\}_{(3)} =$$
$$= \delta_{X_2,X_3}\delta_{B,C}Y^{(1)}_{(A,X_0-X_2)}Y^{(1)}_{(B,X_1-X_3)}Y^{(1)}_{(B,Y_0-Y_1)}. \tag{86}$$

with

$$\Gamma^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)} = \frac{q^{2Z_0-2Z_1+2Y_0-2Y_1}(1)^2_\infty(1-q^2)[d_B]}{(2X_1+1)_1(2Y_0+1)_1(2Z_0+1)_1}\times$$
$$\times\frac{\xi(2X_1+1,2Y_1+1,2Z_1+1)}{\xi(Y_1+X_1-Z_1+1,X_1+Z_1-Y_1+1,Y_1+Z_1-X_1+1,X_1+Y_1+Z_1+2)} \tag{87}$$

Proof:

Straightforward using a Racah-Wigner symmetry on $6j(3)$ and an orthogonality relation on $6j(1)$ symbols. □

Theorem 5 *Using the notations (102) we have*

$$\sqrt{\mathcal{P}(Z_0Z_1)}\sqrt{\Gamma^{(X_0X_1)(Y_0Y_1)}_{(Z_0Z_1)}}\left\{\begin{matrix}B & Y_1\\ Z_1 & X_2\end{matrix}\middle|\begin{matrix}Y_0\\ X_1\end{matrix}\right\}_{(3)} = \sqrt{\hbar}\sqrt{\frac{w^{(AW)}(z;a,b,c,d)}{h_n^{(AW)}(a,b,c,d)}}p_n^{(AW)}(\tau(z);a,b,c,d) \tag{88}$$

with $n = B+Y_0-Y_1, a = q^{2X_1+2Y_1+3}, b = q^{2Y_1-2X_1+1}, c = q^{2X_2-2Y_0+1}, d = q^{-2Y_0-2X_2-1}, 2z = 2Z_1+1$.

Proof:

Using two Sears transformations we can turn the parameters of the hypergeometric functions of the left handside into those of the right handside. Checking now the identity (88) is therefore reduced to straightforward but tedious manipulations on ν_∞ functions. $\square$

Note that the relation (86) when $A = 0$ reduces to the identity:

$$\int d\rho_Z \, \mathcal{P}(0,\rho_Z) \, \Gamma^{(X_0X_1)\,(Y_0Y_1)}_{(Z_0Z_0)} \left\{ \begin{matrix} B & Y_1 \\ Z_0 & X_0 \end{matrix} \middle| \begin{matrix} Y_0 \\ X_1 \end{matrix} \right\}_{(3)} \left\{ \begin{matrix} C & Y_1 \\ Z_0 & X_0 \end{matrix} \middle| \begin{matrix} Y_0 \\ X_1 \end{matrix} \right\}_{(3)} = \delta_{B,C} \tag{89}$$

which is exactly, using (88), the orthogonality condition on Askey-Wilson polynomials for the family of parameters $n = B + Y_0 - Y_1, a = q^{2X_1+2Y_1+3}, b = q^{2Y_1-2X_1+1}, c = q^{2X_0-2Y_0+1}, d = q^{-2Y_0-2X_0-1}, 2z = 2Z_0 + 1$.

Although one would suspect that the relation (86) can be proved for any A by disentangling the sum over m_Z and the integration over ρ_Z, we have not been able to prove it in this way.

6 Conclusion

In this work we have given exact formulae for the Clebsch-Gordan coefficients of the tensor product of two principal unitary representations of the quantum Lorentz Group. We have found explicit expressions of the intertwiners in terms of q-Racah polynomials and Askey-Wilson polynomials. A consequence of this relation is the proof of the proposition (14) which contains in its simplest case the proof of orthogonality of Askey-Wilson polynomials. One should generalize our work to obtain explicit expressions for intertwiners of two representations in the set of principal and complementary series. This should as well gives non trivial relations on orthogonal polynomials.

A very interesting question is the generalization to other quantization of complex semi-simple Lie algebras. In the classical case the structure of the tensor product of two principal representations is known [11] and there also exists polynomials in several variables generalizing Askey-Wilson polynomials [23].

Results [2] which are nevertheless at hand are the construction of 6j(6) in terms of non terminating basic hypergeometric functions, the study of their properties and their understanding as matrix elements of fusion matrices. Using these $6j(6)$, we will directly obtain expressions for the 6j of the principal representations of the quantum Lorentz group as well as very interesting pentagonal equations satisfied by them.

7 Appendix

7.1 Formulae on Basic Hypergeometric Functions.

In the sequel we shall frequently use the following notations:

$\forall x \in \mathbb{C}, \forall k \in \mathbb{N}$,

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} \ , \quad d_x = 2x + 1 \ , \quad [x]_k = \prod_{n=1}^{k} [x + n - 1] \ , \quad [k]! = [1]_k, \quad v_x^{1/4} = e^{i\frac{\pi}{2}x} q^{-\frac{1}{2}x(x+1)}.$$

In this article, the square root of a complex number is defined by:

$$\forall x \in \mathbb{C}, \sqrt{x} = \sqrt{|x|} e^{i\frac{Arg(x)}{2}}, \text{where } \ x = \ |x| e^{iArg(x)}, Arg(x) \in] - \pi, \pi], \tag{90}$$

and for all complex number z with non zero imaginary part, we define $\epsilon(z) = \text{ sign } (Im(z))$.

We will define the following basic functions: $\forall \alpha, \beta, \gamma \in \mathbb{C}, \forall n \in \mathbb{Z}$,

$$(\alpha)_\infty = (q^{2\alpha}, q^2)_\infty = \prod_{k=0}^{+\infty}(1 - q^{2\alpha+2k}) \ , \qquad (\alpha)_n = \frac{(\alpha)_\infty}{(\alpha+n)_\infty},$$

$$\nu_\infty(\alpha) = \prod_{k=0}^{+\infty}\sqrt{1-q^{2\alpha+2k}} \ , \qquad \nu_n(\alpha) = \frac{\nu_\infty(\alpha)}{\nu_\infty(\alpha+n)},$$

$$\omega(\alpha;\beta,\gamma) = \frac{\nu_\infty(\alpha+\beta+\gamma+2)\nu_\infty(\alpha+\beta-\gamma+1)\nu_\infty(\alpha-\beta+\gamma+1)}{\nu_\infty(-\alpha+\beta+\gamma+1)}.$$

The q-factorials satisfy the following relations: $\forall \alpha \in \mathbb{C}, \forall n \in \mathbb{Z}$,

$$(\alpha)_{-n}(1-\alpha)_n = (-1)^n q^{-2n\alpha+n(n+1)} \ \text{(inversion relation)}, \tag{91}$$

$$\lim_{\alpha\to+\infty}(\alpha)_n = 1, \qquad (\alpha)_n \sim_{\alpha\to-\infty} (-1)^n q^{2n\alpha+n(n-1)}. \tag{92}$$

In order to control the signs of the expressions in our article, we are led to introduce the following functions $\varphi : \mathbb{C} \times \frac{1}{2}\mathbb{Z} \to \mathbb{C}$, defined by:

$$\varphi_{(\alpha,n)} = \frac{\nu_\infty(\alpha-n+1)\nu_\infty(n-\alpha)}{\nu_\infty(\alpha+1)\nu_\infty(-\alpha)} q^{-n\alpha+\frac{1}{2}n(n-1)} \ , \forall \alpha \in \mathbb{C}, \forall n \in \frac{1}{2}\mathbb{Z}. \tag{93}$$

This function φ satisfies the two following relations:

$$\varphi_{(\alpha,n)}\varphi_{(\alpha-n,p)} = \varphi_{(\alpha,n+p)}, \forall \alpha \in \mathbb{C}, \forall n, p \in \frac{1}{2}\mathbb{Z} \tag{94}$$

$$\frac{\varphi^2_{(\alpha,n)}}{\varphi^2_{(\alpha,p)}} = (-1)^{n-p}, \forall \alpha \in \mathbb{C}, \forall n, p \in \frac{1}{2}\mathbb{Z} \ , (n-p) \in \mathbb{Z}, \tag{95}$$

The last equation implies in particular that $\varphi^2_{(\alpha,n)} = (-1)^n, \forall \alpha \in \mathbb{C}, \forall n \in \mathbb{Z}$. Note that when n is half an odd integer, $\varphi_{(\alpha,n)}$ is not a fourth root of unity.

With our choice of square root, we have the more precise results:

$$\varphi_{(\alpha,n)} = e^{-i\frac{\pi}{2}\ n\ \epsilon(q^\alpha)}, \forall n \in \mathbb{Z}, \forall \alpha \in \mathbb{C}, \tag{96}$$

$$|\varphi_{(-1+i\rho,n)}|^2 = e^{i\pi(p+1/2+\epsilon(q^{i\rho}))}\varphi^2_{(i\rho,-p)}, \forall n, p \in \frac{1}{2} + \mathbb{Z} \ , \forall \rho \in \mathbb{R}. \tag{97}$$

We will also make an extensive use of the following basic hypergeometric functions, associated to complex numbers $\alpha_0, \cdots, \alpha_n, \beta_1, \cdots, \beta_n$ and defined by:

$$\forall Z \in \mathbb{C}, \qquad {}_{n+1}\Phi_n \left[\begin{matrix} \alpha_0 & \alpha_1 & \cdots & \alpha_n & \\ & \beta_1 & \cdots & \beta_n & ; Z \end{matrix} \right] = \sum_{k=0}^{+\infty} \frac{\prod_{i=0}^n (\alpha_i)_k}{(1)_k \prod_{i=1}^n (\beta_i)_k} q^{2kZ},$$

$${}_{n+3}W_{n+2}(\alpha_0 \ ; \ \alpha_1, \cdots, \alpha_n \ ; \ Z) = \sum_{k=0}^{+\infty} \frac{(1-q^{2\alpha_0+4k})}{(1-q^{2\alpha_0})} \frac{\prod_{i=0}^n (\alpha_i)_k}{(1)_k \prod_{i=1}^n (1+\alpha_0-\alpha_i)_k} q^{2kZ}.$$

In an expression involving ${}_{n+1}\Phi_n$ or ${}_{n+1}W_n$, if Z is not specified it will mean that $Z = 1$.

Let us recall some properties of basic hypergeometric functions which are proved for example in [24]:

$$ {}_2\Phi_1\left[\begin{matrix}\alpha_0 & \alpha_1, & \\ & \beta_1 & ;\beta_1-\alpha_0-\alpha_1\end{matrix}\right]=\frac{(\beta_1-\alpha_0)_\infty(\beta_1-\alpha_1)_\infty}{(\beta_1)_\infty(\beta_1-\alpha_0-\alpha_1)_\infty}, \tag{98}$$

$$ {}_4\Phi_3\left[\begin{matrix}-n & \alpha_1 & \alpha_2 & \alpha_3\\ & \beta_1 & \beta_2 & \beta_3\end{matrix}\right]=\frac{{}_4\Phi_3\left[\begin{matrix}-n & \alpha_1 & \beta_1-\alpha_2 & \beta_1-\alpha_3\\ & \beta_1 & 1-n+\alpha_1-\beta_2 & 1-n+\alpha_1-\beta_3\end{matrix}\right]}{q^{2n\alpha_1}(\beta_2-\alpha_1)_n^{-1}(\beta_3-\alpha_1)_n^{-1}(\beta_2)_n(\beta_3)_n}, \tag{99}$$

$$ {}_6W_5(\alpha_0\ ;\ \alpha_1,\alpha_2,-n\ ;\ 1+n+\alpha_0-\alpha_1-\alpha_2)=\frac{(1+\alpha_0)_\infty(1+\alpha_0-\alpha_1-\alpha_2)_\infty}{(1+\alpha_0-\alpha_1)_\infty(1+\alpha_0-\alpha_2)_\infty}. \tag{100}$$

Relation (98) is called Heine formula, relation (99) is the Sears Identity.

Let us now recall the definitions of $q-$Racah polynomials $p_n^{(R)}$, of the $q-$Racah discrete measure $w^{(R)}$, and of the square of the norm of these polynomials [19][20]:

$$\begin{aligned}
&\forall x\in\mathbb{C},\forall\alpha,\beta,\gamma,\delta\in\mathbb{C},\mu(x)=q^{-2x}+q^{2x+2+2\gamma+2\delta},\\
&p_n^{(R)}(\mu(x);q^{2\alpha},q^{2\beta},q^{2\gamma},q^{2\delta})={}_4\Phi_3\left[\begin{matrix}-n & n+1+\alpha+\beta & -x & \gamma+\delta+x+1\\ & \alpha+1 & \beta+\delta+1 & \gamma+1\end{matrix}\right],\\
&w^{(R)}(x;q^{2\alpha},q^{2\beta},q^{2\gamma},q^{2\delta})=\frac{(\gamma+\delta+1,\alpha+1,\beta+\delta+1,\gamma+1)_x(1-q^{2(2x+\gamma+\delta+1)})}{q^{2x(\alpha+\beta+1)}(1,\gamma+\delta+1-\alpha,\gamma-\beta+1,\delta+1)_x(1-q^{2(\gamma+\delta+1)})},\\
&h_n^{(R)}(q^{2\alpha},q^{2\beta},q^{2\gamma},q^{2\delta})=\frac{q^{2n(\gamma+\delta+1)}(1-q^{2(\alpha+\beta+1)})(1,\beta+1,\alpha-\delta+1,\alpha+\beta-\gamma+1)_n}{(1-q^{2(2n+\alpha+\beta+1)})(\alpha+\beta+1,\alpha+1,\beta+\delta+1,\gamma+1)_n}\times\\
&\qquad\times\frac{(\gamma+\delta+2,\gamma-\alpha-\beta,\delta-\alpha,-\beta)_\infty}{(\gamma+\delta-\alpha+1,\gamma-\beta+1,\delta+1,-\alpha-\beta-1)_\infty}.
\end{aligned}\tag{101}$$

The definition of Askey-Wilson polynomials $p_n^{(AW)}$, of the Askey-Wilson measure $w^{(AW)}$ and of the square of the norm of these polynomials is [21]:

$$\begin{aligned}
&\forall z\in\mathbb{C},\forall\alpha,\beta,\gamma,\delta\in\mathbb{C},\tau(z)=\frac{1}{2}(q^{2z}+q^{-2z})\\
&\frac{p_n^{(AW)}(\tau(z);q^{2\alpha},q^{2\beta},q^{2\gamma},q^{2\delta})}{q^{-2n\alpha}(\alpha+\beta,\alpha+\gamma,\alpha+\delta)_n}={}_4\Phi_3\left[\begin{matrix}-n & n+\alpha+\beta+\gamma+\delta-1 & \alpha+z & \alpha-z\\ & \alpha+\beta & \alpha+\gamma & \alpha+\delta\end{matrix}\right],\\
&w^{(AW)}(z;q^{2\alpha},q^{2\beta},q^{2\gamma},q^{2\delta})=\frac{(2z,-2z)_\infty}{(\alpha+z,\alpha-z,\beta+z,\beta-z,\gamma+z,\gamma-z,\delta+z,\delta-z)_\infty},\\
&h_n^{(AW)}(q^{2\alpha},q^{2\beta},q^{2\gamma},q^{2\delta})=\frac{(\alpha+\beta+n)_\infty^{-1}(n+1)_\infty^{-1}(\alpha+\beta+\gamma+\delta+2n)_\infty(\alpha+\beta+\gamma+\delta+n-1)_n}{(\alpha+\gamma+n,\alpha+\delta+n,\beta+\gamma+n,\beta+\delta+n,\gamma+\delta+n)_\infty}.
\end{aligned}\tag{102}$$

7.2 Asymptotics of Intertwiners

Our aim is to compute the behaviour of $\left[\begin{matrix}X_0X_1 & Y_0Y_1\\ P+T & P\end{matrix}\middle|\begin{matrix}T\\ Z_0Z_1\end{matrix}\right]$ (see (63) for the explicit expression) for $P=Y_0-Y_1\geq 0, T\to+\infty$. The sum on X_2 being finite, we can work term by term. The asymptotics of $\left\{\begin{matrix}T & P\\ X_1 & X_0\end{matrix}\middle|\begin{matrix}P+T\\ X_2\end{matrix}\right\}_{(1)}$ is very easy to obtain because, after a Racah-Wigner symetry (24), the hypergeometric part of the result tends to one when T is large. The behaviour is therefore given by:

$$\left\{\begin{matrix}T & P\\ X_1 & X_0\end{matrix}\middle|\begin{matrix}P+T\\ X_2\end{matrix}\right\}_{(1)}\sim\frac{\varphi_{(X_0+X_2-T,P+X_2-X_1)}\nu_1(2X_2+1)\omega(P;X_1,X_2)}{q^{\frac{1}{2}(P-3X_2-X_1-1)(P+X_2-X_1)}\nu_\infty(2P+1)\nu_\infty(1)}$$

The behaviour of $\left\{\begin{matrix} T & Z_1 \\ Y_0 & X_0 \end{matrix}\middle| \begin{matrix} Z_0 \\ X_2 \end{matrix}\right\}_{(3)}$ requires more attention, but can be achieved using the following computation.

By using first the Sears identity (99), and then taking the limit when T goes to infinity, we easily conclude using Heine summation formula (98):

$$\frac{(Z_1+Y_0-X_0-T+1)_\infty {}_4\Phi_3\begin{bmatrix} Z_1-Z_0-T & Z_1+Z_0-T+1 & X_2-X_0-T & -X_2-X_0-T-1 \\ -2T & -X_0-Y_0-T+Z_1 & Z_1+Y_0-X_0-T+1 & \end{bmatrix}}{(Z_0+Y_0-X_0+1, Y_0-X_0-Z_0, 2Z_1+2)_\infty} =$$

$$= \frac{(2T+1)_\infty {}_4\Phi_3\begin{bmatrix} Z_1-Z_0-T & Z_1+Z_0-T+1 & Z_1-X_2-Y_0 & X_2+Z_1-Y_0+1 \\ 2Z_1+2 & -X_0-Y_0-T+Z_1 & Z_1+X_0-Y_0-T+1 & \end{bmatrix}}{(T+Y_0-Z_1-X_0, T+Z_1+Z_0+2, Z_1+T-Z_0+1)_\infty}$$

$$\sim_{T\to+\infty} {}_2\Phi_1\begin{bmatrix} Z_1-X_2-Y_0 & X_2+Z_1-Y_0+1 & \\ & 2Z_1+2 & ;2Y_0+1 \end{bmatrix}$$

$$\sim_{T\to+\infty} \frac{(Z_1+Y_0+X_2+2, Z_1+Y_0-X_2+1)_\infty}{(2Y_0+1, 2Z_1+2)_\infty}.$$

This gives us the following behaviour

$$\left\{\begin{matrix} T & Z_1 \\ Y_0 & X_0 \end{matrix}\middle| \begin{matrix} Z_0 \\ X_2 \end{matrix}\right\}_{(3)} \sim e^{i\pi(T+X_0-X_2)} q^{-2TY_0} \frac{\varphi(X_0+X_2-T,-T)\varphi(Z_0+Z_1-T,-T)\omega(Y_0;X_2,Z_1)\omega(Y_0;X_0,Z_0)}{q^{(Z_1-Z_0)(X_0+Z_0-Y_0)+(X_2-X_0)(Z_1+X_0-Y_0+1)}(1,2Y_0+1)_\infty} \times$$
$$\times \frac{\nu_1(2X_2+1)\nu_1(2Z_0+1)(Y_0-X_0-Z_0, X_0+Z_0-Y_0+1)_\infty}{\nu_\infty(-X_0-X_2)\nu_\infty(X_0+X_2+1)\nu_\infty(-Z_0-Z_1)\nu_\infty(Z_0+Z_1+1)} \tag{103}$$

The expression of $\left\{\begin{matrix} P & Y_1 \\ Z_1 & X_2 \end{matrix}\middle| \begin{matrix} Y_0 \\ X_1 \end{matrix}\right\}_{(3)}$ being drastically simplified for $P = Y_0 - Y_1$, there is no problem anymore to obtain the behaviour of the whole expression, except that in order to have a simple answer, we have to take the sum over X_2. Precisely, denoting $\alpha = 2X_1+2Y_1-2Y_0+1$, $m = 2Y_0-2Y_1$, $k = Y_0-Y_1+X_2-X_1$ we already have obtained the following behaviour when $T \to +\infty$

$$\left[\begin{matrix} X_0X_1 & Y_0Y_1 \\ T+Y_0-Y_1 & Y_0-Y_1 \end{matrix}\middle| \begin{matrix} T \\ Z_0Z_1 \end{matrix}\right] \sim q^{-T(Y_0+Y_1)}\ e^{i\pi(\frac{X_0-X_1}{2})} q^{-Z_0Z_1} \times$$
$$\times \sqrt{[d_{Y_0-Y_1}]}\nu_1(2X_1+1)\nu_1(2Z_0+1)\frac{\varphi(2X_0, P+T+X_0-X_1)\varphi(Z_1+X_1-Y_1, Z_1-Z_0+X_1-X_0+Y_0-Y_1)}{\varphi(Z_0+Z_1,T)} \times$$
$$\times \frac{\nu_\infty(Y_0+X_0-Z_0+1)\nu_\infty(Y_0+X_0+Z_0+2)\nu_\infty(Y_0-X_0-Z_0)\nu_\infty(Y_0+Z_0-X_0+1)}{q^{\frac{3}{2}X_0^2+\frac{1}{2}X_0-\frac{1}{2}Z_0^2+\frac{1}{2}Z_0}(1)_\infty\nu_\infty(2Y_1+1)\nu_\infty(2Y_0+1)} \times$$
$$\times \frac{\nu_\infty(Y_1+X_1-Z_1+1)\nu_\infty(Y_1+X_1+Z_1+2)\nu_\infty(Y_1-X_1-Z_1)\nu_\infty(Y_1+Z_1-X_1+1)}{q^{-\frac{3}{2}X_1^2-\frac{1}{2}X_1-\frac{1}{2}Z_1^2-\frac{1}{2}Z_1}\nu_\infty(2X_0+1)\nu_\infty(-2X_0)\nu_\infty(Z_0+Z_1+1)\nu_\infty(-Z_0-Z_1)} \times A \tag{104}$$

where
$A = \frac{1}{(1+\alpha)_m}\sum_k \frac{(1-q^{2\alpha+4k})(\alpha)_k(-m)_k}{(1-q^{2\alpha})(1)_k(\alpha+m+1)_k} q^{2k^2+2k(m+\alpha)}$.

Using the formula (100) and taking in it the limit where $\alpha_1, \alpha_2 \to -\infty$, we obtain that $A = 1$. This gives the behaviour of $\left[\begin{matrix} X_0X_1 & Y_0Y_1 \\ T+Y_0-Y_1 & Y_0-Y_1 \end{matrix}\middle| \begin{matrix} T \\ Z_0Z_1 \end{matrix}\right]$ when T goes to infinity.

7.3 Formulae on coefficients $\Lambda_{AD}^{BC}(X_0X_1)$

Let us recall some properties of the coefficients $\Lambda_{AD}^{BC}(X_0X_1)$ which were proved in [1]. We will say that the coefficients Λ_{AD}^{BC} are off-diagonal if $A \neq D$ and diagonal if $A = D$ (in the latter case

we will use the notation $\Lambda_A^{BC} = \Lambda_{AA}^{BC}$). The coefficients Λ_A^{BC} are said to be on the boundary if $B+C=A$, $A+B=C$, or $A+C=B$. The coefficients $\Lambda_{AD}^{BC}(X_0X_1)$ are proportional to $Y^{(0)}_{(A,B,C)}Y^{(0)}_{(D,B,C)}Y^{(1)}_{(A,X_0-X_1)}Y^{(1)}_{(D,X_0-X_1)}$, so they vanish according to these selection rules. The following relation holds true (it is in fact equivalent to the property that $\overset{(X_0X_1)}{\Pi}$ is a representation of $\mathfrak{U}_q(an(2))$):

$$\Lambda_{FG}^{AC}\Lambda_{GH}^{BD} = \sum_{KU} \frac{e^{i\pi(G+U)}([d_U][d_G])^{\frac{1}{2}}\ \Lambda_{FH}^{KU}}{e^{i\pi(C+D)}([d_C][d_D])^{\frac{1}{2}}} \left\{\begin{matrix} B & C \\ A & D \end{matrix}\middle|\begin{matrix} U \\ G \end{matrix}\right\}_{(0)} \left\{\begin{matrix} F & A \\ B & U \end{matrix}\middle|\begin{matrix} C \\ K \end{matrix}\right\}_{(0)} \left\{\begin{matrix} A & B \\ H & U \end{matrix}\middle|\begin{matrix} K \\ D \end{matrix}\right\}_{(0)}. \quad (105)$$

As a result, by taking $F=H$ in this expression, the off-diagonal coefficients are obtained, up to a sign, from the diagonal coefficients.

These diagonal coefficients can be computed in various ways. They satisfy the following system of linear equations:

$$\forall (A,B,C) \in \frac{1}{2}\mathbb{Z}^+ \times \frac{1}{2}\mathbb{Z}^+ \times \frac{1}{2}\mathbb{Z}^{+*} \ / \ B+A-C,\ B+C-A,\ A+C-B \in \mathbb{N}^*$$

$$\frac{[A+B+C+1]}{[2A+1]}\Lambda^{B\ C}_{\ A} + \frac{[A+C-B+1]}{[2A+1]}\Lambda^{B-1\ C}_{\ \ A} - \frac{[C+B-A-1]}{[2A+1]}\Lambda^{B-1\ C-1}_{\ \ A} + \frac{[B+A-C+1]}{[2A+1]}\Lambda^{B\ C-1}_{\ A} =$$
$$= \frac{[A+B-C-1]}{[2A-1]}\Lambda^{B-1\ C}_{\ A-1} + \frac{[A+C-B-1]}{[2A-1]}\Lambda^{B\ C-1}_{\ A-1} + \frac{[A+B+C-1]}{[2A-1]}\Lambda^{B-1\ C-1}_{\ A-1} - \frac{[C+B-A+1]}{[2A-1]}\Lambda^{B\ C}_{\ A-1}, \quad (106)$$

as well as $\forall A,B,C \in \frac{1}{2}\mathbb{Z}^+ / Y^{(0)}_{(A,B,C)} = 1$,

$$[B+C-A+1][A+B+C+2]q^{\pm(C-B)}\Lambda^{B+\frac{1}{2}\ C+\frac{1}{2}}_{\ \ A} - [A+C-B][A+B-C+1]q^{\mp(C+B+1)}\Lambda^{B+\frac{1}{2}\ C-\frac{1}{2}}_{\ \ A} +$$
$$+[A+C+B+1][B+C-A]q^{\mp(C-B)}\Lambda^{B-\frac{1}{2}\ C-\frac{1}{2}}_{\ \ A} - [A+B-C][A+C-B+1]q^{\pm(C+B+1)}\Lambda^{B-\frac{1}{2}\ C+\frac{1}{2}}_{\ \ A}$$
$$= \omega_\pm[2C+1][2B+1]\Lambda^{B\ C}_{\ A}, \quad (107)$$

where we have denoted $\omega_+ = q^{2X_0+1}+q^{-2X_0-1}$, $\omega_- = q^{2X_1+1}+q^{-2X_1-1}$.

These two linear systems have a unique normalized solution. The value of the diagonal coefficients on the boundary have the following simple expression:

$$\Lambda^{B\ C}_{B+C}(X_0X_1) = \sum_{k=-B}^{B} \frac{q^{-2k(X_0+X_1+1)}\binom{B+C+X_0-X_1}{B+k}_q\binom{B+C-X_0+X_1}{B-k}_q}{(-1)^{2B}\binom{2B+2C}{2B}_q}, \quad (108)$$

$$\Lambda^{B\ A+B}_{\ A}(X_0X_1) = \sum_{k=-B}^{B} \frac{q^{-2k(X_0+X_1+1)}\binom{A+X_0-X_1+B-k}{B-k}_q\binom{A-X_0+X_1+B+k}{B+k}_q}{\binom{2A+2B+1}{2B}_q}$$
$$= \frac{v_{X_0}}{v_{X_1}}\Lambda^{A+B\ B}_{\ \ A}(X_1X_0). \quad (109)$$

Different explicit formulae for the coefficients Λ_{AD}^{BC} can be obtained, the fundamental one is expressed in terms of 6j(1) (53). This last expression can be simplified, using the universal shifted cocycle associated to $6j(1)$, and we get:

$$\Lambda_{AD}^{BC}(X_0,X_1) = \sum_{\sigma_1\sigma_2} \frac{v_A^{1/4}}{v_D^{1/4}}\frac{\mathcal{N}^{(D)}(q^{2X_1+1},m_X)}{\mathcal{N}^{(A)}(q^{2X_1+1},m_X)} \begin{pmatrix} m_X & C & B \\ A & \sigma_2 & \sigma_1 \end{pmatrix} q^{-2i\sigma_1\rho_X} \begin{pmatrix} \sigma_1 & \sigma_2 & D \\ B & C & m_X \end{pmatrix} \quad (110)$$

where $\mathcal{N}^{(A)}(q^{2X+1},\sigma)$ are normalization factors, given in [1], which values will not be useful in our paper. This last expression is simpler than (53) in many aspects: it is a sum of a product

of basic hypergeometric of 3-2 type rather than 4-3 type, and moreover it explicitely shows that Λ_A^{BC} are Laurent polynomials in $q^{i\rho x}$.

Remark: From the system of constraints satisfied by the coefficients Λ_{AD}^{BC}, it is easy to show that $(\Lambda_{AD}^{BC})^2$ is a Laurent polynomials in both variables q^{2X_0+1}, q^{2X_1+1}.

The constraint systems described before, as well as explicit expressions, allow us to derive most of the asymptotic properties of these coefficients when some of the variables are large.

Proposition 15 *The coefficients Λ_{AD}^{BC} satisfy the following inequality:*

$$\forall A, D \in \frac{1}{2}\mathbb{Z}^+, \forall R \in \frac{1}{2}\mathbb{Z}, \forall m \in \frac{1}{2}\mathbb{Z}, \exists\, \mathcal{C}, \mathcal{B} > 0, \forall \rho \in \mathbb{R}, \forall B > \mathcal{B}, |\Lambda_{A\ \ D}^{B\ B+R}(m,\rho)| \leq \mathcal{C} B q^{2B}. \tag{111}$$

Proof:
The proof is divided in two steps. We first show this inequality when $A = D$, by a direct computation. Then we use the system (105) and an induction argument to show this inequality when $A \neq D$. We can always assume that $R \geq 0$, because the other case is deduced from this one and the identity (109). Using the relation (110), we have $|\Lambda_A^{BB+R}(m,\rho)| \leq \sum_j u(j, m-j; B)$, where $u(j,k;B) = |\begin{pmatrix} m & B+R & B \\ A & j & k \end{pmatrix} \begin{pmatrix} -m & B+R & B \\ A & -j & -k \end{pmatrix}|$.
Using the formula (4) of ([17], section 14.3.5), we easily obtain that:

$$\frac{|u(j,m-j;B)|}{q^{(A-R)(2B+R+A+1)}} = \frac{[2A]![2A+1]![A+B-j]![B+A+j]![2B+R-A]!F(j,m-j;B)F(-j,j-m;B)}{[A+R]![A-R]![A+m]![A-m]![B+m-j]![B-m+j]![2B+R+A+1]!}$$

where $F(j,m-j;B) = {}_3\Phi_2 \begin{bmatrix} m-A & -2B-R-A-1 & R-A \\ & -B-A+j & -2A \end{bmatrix}$. Using the fact that $-2B-A-R \leq -B-A+j \leq R-A$, we can choose C_1 such that $\forall j, \forall B, |F(j,m-j;B)| \leq C_1 q^{-2Bn(m)}$ where $n(m) = inf(A-m, A-R)$. As a result, there exists a constant C_1 such that

$$\forall j, B, \qquad u(j,m-j;B) \leq C_1 q^{2B(A-R-n(m)-n(-m))-|B+j|(A+m)-|B-j|(A-m)-2B(2A+1)}.$$

From this, and the various inequalities between B, j, m, R, it is easy to show that we can find C_2 such that

$$|\Lambda_A^{BB+R}(m,\rho)| \leq C_2(2B+2R+1)q^{2B}. \tag{112}$$

We can now prove this bound for the non diagonal case using an induction argument on $|A-D|$. Indeed from (105),

$$|\Lambda_{AD+1}^{BB+R}\Lambda_{D+1D}^{\frac{1}{2}D+\frac{1}{2}}| \leq \sum_{KU} \frac{([d_U][d_{D+1}])^{\frac{1}{2}}|\Lambda_{AD}^{KU}|}{([d_{B+R}][d_{D+\frac{1}{2}}])^{\frac{1}{2}}} |\left\{\begin{matrix} \frac{1}{2} & B+R \\ A & D+\frac{1}{2}\end{matrix}\middle|\begin{matrix} U \\ D+1\end{matrix}\right\}_{(0)}\left\{\begin{matrix} A & B \\ \frac{1}{2} & U\end{matrix}\middle|\begin{matrix} B+R \\ K\end{matrix}\right\}_{(0)}\left\{\begin{matrix} B & \frac{1}{2} \\ D & U\end{matrix}\middle|\begin{matrix} K \\ D+\frac{1}{2}\end{matrix}\right\}_{(0)}|.$$

Note that, due to the Y functions, the previous sums contain at most four terms: $U = B+R\pm\frac{1}{2}, K = B\pm\frac{1}{2}$. From the asymptotics property (112) when B goes to $+\infty$, and the induction hypothesis, we obtain that there exists a constant C_3, with $\forall B, \forall \rho, |\Lambda_{AD+1}^{BB+R}(m,\rho)\Lambda_{D+1D}^{\frac{1}{2}D+\frac{1}{2}}(m,\rho)| \leq C_3 B q^{2B}$. From the explicit formulae of the coefficients $\Lambda_{D+1D}^{\frac{1}{2}D+\frac{1}{2}}(m,\rho)$, in [1], there exists $C_4 > 0$ such that $\forall \rho \in \mathbb{R}, \forall B, |\Lambda_{D+1D}^{\frac{1}{2}D+\frac{1}{2}}(m,\rho),| > C_4$, as a result we get that the induction hypothesis is also true for $A, D+1$. $\square$

We will also need the following asymptotics:

Proposition 16 *The coefficients Λ_A^{BC} own the following asymptotic behaviour:*

$$\forall B \in \frac{1}{2}\mathbb{Z}^+, \forall R \in \frac{1}{2}\mathbb{Z}, B \pm R \in \mathbb{N}, \forall (X_0X_1) \in \mathbb{S}, \exists \mathcal{C} \in \mathbb{R}$$
$$\Lambda^{B\ \ K+R}_{\ K} \sim_{K\to+\infty} q^{2K|R|}\, \mathcal{C}\, Y^{(1)}_{(B,R)} \quad \textit{with}\ \ (\ \mathcal{C} = 1\ \ \textit{when}\ \ R = 0). \tag{113}$$

Proof:
We use the formula (110) to express Λ_K^{BK+R}. The asymptotic when K is large is easily obtained using the formula (1) of ([17] section 14.3.5). The details are left to the reader and we easily obtain the statement of the proposition. □

In the section 5 we will need the following identity on the following series of weighted diagonal coefficients:

Proposition 17 *If $P > 0$ the following series is absolutely and uniformly convergent in ρ_X and we have:*

$$\sum_{KM}[d_K]\begin{pmatrix} \sigma-P & K & P \\ M & \sigma & -P \end{pmatrix}\begin{pmatrix} -P & \sigma & M \\ P & K & \sigma-P \end{pmatrix}\Lambda_P^{KM}(X_0X_1) = \frac{\hbar[d_P]q^{-2i\sigma\rho_X}e^{i\pi\sigma}\delta_{m_X,P}}{2\pi\mathcal{P}(X_0X_1)}. \tag{114}$$

When $P = 0$ we still have :

$$\sum_K [d_K]\Lambda_0^{KK}(X_0X_1) = \frac{\hbar[d_P]\delta_{m_X,P}}{2\pi\mathcal{P}(X_0X_1)}. \tag{115}$$

Proof:
When $P = 0$, we see from (110) that $\Lambda_0^{KK}(X_0X_1)$ is non zero only when $m_X = 0$. The series $\sum_K [d_K]\Lambda_0^{KK}(X_0X_1)$ is in this case very simple to compute because $\Lambda_0^{KK}(X_0X_1) = \frac{[(2X_0+1)(2K+1)]}{[2X_0+1][2K+1]}$, and we obtain the announced statement. Note that in this case the series is not uniformly convergent and is even divergent at $\rho_Z = 0$. When $P > 0$ we will first show that the series is absolutely and uniformly convergent. Using the analogue of the Van-Der-Waerden formula for $3j$ [17], as well as relation (10) we can choose C_1 such that:

$$\begin{aligned} \left|\begin{pmatrix} \sigma-P & K & P \\ K+R & \sigma & -P \end{pmatrix}\begin{pmatrix} -P & \sigma & K+R \\ P & K & \sigma-P \end{pmatrix}\right| &= \left|\frac{[d_{K+R}][2P]![2K+R-P]![K+R+P-\sigma]![K+\sigma]!}{[P-R]![R+P]![2K+R+P+1]![K+R-P+\sigma]![K-\sigma]!}\right| \\ &\leq\ C_1\, q^{2KP} \end{aligned}$$

Using $\Lambda_P^{KK+R}(X_0X_1) \leq C_2\ Kq^{2K}$, we obtain that the series is absolutely and uniformly convergent. Using (105)(59)(7)(110) we have

$$\int d\mathcal{P}(X_0X_1)\left(\sum_{KM}[d_K]\begin{pmatrix} \sigma-P & K & P \\ M & \sigma & -P \end{pmatrix}\begin{pmatrix} -P & \sigma & M \\ P & K & \sigma-P \end{pmatrix}\Lambda_P^{KM}(X_0X_1)\right)\Lambda_P^{AC}(X_0X_1) =$$

$$= \sum_{KM}[d_K]\begin{pmatrix} \sigma-P & K & P \\ M & \sigma & -P \end{pmatrix}\begin{pmatrix} -P & \sigma & M \\ P & K & \sigma-P \end{pmatrix}\int d\mathcal{P}(X_0X_1)\Lambda_P^{KM}(X_0X_1)\Lambda_P^{AC}(X_0X_1) =$$

$$= \sum_{KM}[d_K]\begin{pmatrix} \sigma-P & K & P \\ M & \sigma & -P \end{pmatrix}\begin{pmatrix} -P & \sigma & M \\ P & K & \sigma-P \end{pmatrix}\sum_{JU}\frac{([2U+1][2P+1])^{\frac{1}{2}}}{([2C+1][2M+1])^{\frac{1}{2}}e^{i\pi(C+M-P-U)}}\times$$
$$\times\begin{Bmatrix} K & C & U \\ A & M & P \end{Bmatrix}_{(0)}\begin{Bmatrix} P & A & C \\ K & U & J \end{Bmatrix}_{(0)}\begin{Bmatrix} A & K & J \\ P & U & M \end{Bmatrix}_{(0)}\cdot\int d\mathcal{P}(X_0X_1)\Lambda_P^{JU}(X_0X_1) =$$

$$= \sum_{KM} [d_K] \left(\begin{array}{c} \sigma - P \\ M \end{array} \middle| \begin{array}{cc} K & P \\ \sigma & -P \end{array} \right) \left(\begin{array}{cc} -P & \sigma \\ P & K \end{array} \middle| \begin{array}{c} M \\ \sigma - P \end{array} \right) e^{2i\pi A} \delta_{A,K} \frac{[d_P]}{[d_A]} \left\{ \begin{array}{cc} A & C \\ A & M \end{array} \middle| \begin{array}{c} P \\ P \end{array} \right\}_{(0)} =$$

$$= \sum_{M} [d_M] \left(\begin{array}{cc} P - \sigma & \sigma \\ M & A \end{array} \middle| \begin{array}{c} P \\ P \end{array} \right) \left(\begin{array}{c} P \\ P \end{array} \middle| \begin{array}{cc} A & M \\ \sigma & P - \sigma \end{array} \right) \left\{ \begin{array}{cc} A & C \\ A & M \end{array} \middle| \begin{array}{c} P \\ P \end{array} \right\}_{(0)} =$$

$$= \sum_{M} [d_M] \left\{ \begin{array}{cc} A & C \\ A & M \end{array} \middle| \begin{array}{c} P \\ P \end{array} \right\}_{(0)} \frac{\hbar}{2\pi} \int_{-\frac{\pi}{\hbar}}^{\frac{\pi}{\hbar}} d\rho \, q^{2i\sigma\rho} \Lambda_P^{AM}(P, \rho).$$

Then we use the relation

$$\sum_{M} [d_M] \left\{ \begin{array}{cc} A & C \\ A & M \end{array} \middle| \begin{array}{c} P \\ P \end{array} \right\}_{(0)} \Lambda_P^{AM}(X_0 X_1) = e^{-2i\pi A} [d_P] \, \Lambda_P^{AC}(X_1 X_0).$$

which has been proved in [1] and which is the key point to prove unitarity of the representation and the relation

$$\int_{-\frac{\pi}{\hbar}}^{\frac{\pi}{\hbar}} d\rho \, q^{2i\sigma\rho} \Lambda_P^{AC}(-P, \rho) = \int_{-\frac{\pi}{\hbar}}^{\frac{\pi}{\hbar}} d\rho \, q^{-2i\sigma\rho} \Lambda_P^{AC}(P, \rho) \tag{116}$$

which is proved using twice (110),(7) and (59). Combining these two steps, we obtain

$$\int d\mathcal{P}(X_0 X_1) \left(\sum_{KM} [d_K] \left(\begin{array}{c} \sigma - P \\ M \end{array} \middle| \begin{array}{cc} K & P \\ \sigma & -P \end{array} \right) \left(\begin{array}{cc} -P & \sigma \\ P & K \end{array} \middle| \begin{array}{c} M \\ \sigma - P \end{array} \right) \Lambda_P^{KM}(X_0 X_1) \right) \Lambda_P^{AC}(X_0 X_1) =$$

$$= \int d\mathcal{P}(X_0 X_1) \, \frac{\hbar \, e^{2i\pi\sigma} [d_P] q^{-2\sigma(X_0 + X_1 + 1)} \delta_{X_0 - X_1, P}}{2\pi \, \mathcal{P}(X_0 X_1)} \Lambda_P^{AC}(X_0 X_1).$$

Finally using Lemma 1 we conclude this proof. □

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